

# Consistency and Advantage of Loop Regularization Method Merging with Bjorken-Drell's Analogy Between Feynman Diagrams and Electrical Circuits

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## Abstract

The consistency of loop regularization (LORE) method is explored in multiloop calculations. A key concept of the LORE method is the introduction of irreducible loop integrals (ILIs) which are evaluated from the Feynman diagrams by adopting the Feynman parametrization and ultraviolet-divergence-preserving(UVDP) parametrization. It is then inevitable for the ILIs to encounter the divergences in the UVDP parameter space due to the generic overlapping divergences in the 4-dimensional momentum space. By computing the so-called  $\alpha\beta\gamma$  integrals arising from two loop Feynman diagrams, we show how to deal with the divergences in the parameter space with the LORE method. By identifying the divergences in the UVDP parameter space to those in the subdiagrams, we arrive at the Bjorken-Drell's analogy between Feynman diagrams and electrical circuits. The UVDP parameters are shown to correspond to the conductance or resistance in the electrical circuits, and the divergence in Feynman diagrams is ascribed to the infinite conductance or zero resistance. In particular, the sets of conditions required to eliminate the overlapping momentum integrals for obtaining the ILIs are found to be associated with the conservations of electric voltages, and the momentum conservations correspond to the conservations of electrical currents, which are known as the Kirchhoff's laws in the electrical circuits analogy. As a practical application, we carry out a detailed calculation for one-loop and two-loop Feynman diagrams in the massive scalar  $\phi^4$  theory, which enables us to obtain the well-known logarithmic running of the coupling constant and the consistent power-law running of the scalar mass at two loop level. Especially, we present an explicit demonstration on the general procedure of applying the LORE method to the multiloop calculations of Feynman diagrams when merging with the advantage of Bjorken-Drell's circuit analogy.

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## I. INTRODUCTION

Quantum Field Theory(QFT) is the most successful theory for understanding the microscopic world in elementary particle physics, nuclear physics and condensed matter physics. However, when carrying out any calculation beyond tree level in the framework of perturbation treatment of QFT, one would encounter the infinities in Feynman integrals, coming from contribution from large momenta and are usually called ultraviolet (UV) divergences. Thus, the QFT becomes well-defined only when it can be regularized and renormalized properly.

Nevertheless, the widely-used regularization methods are known to have some limitations. For instance, the Pauli-Vallars regularization method is very useful in the calculation of quantum electrodynamics(QED), but it fails in non-Abelian gauge theory as it explicitly destroys the non-Abelian gauge invariance. The dimensional regularization can preserve the gauge symmetry explicitly and has been useful in the computations for gauge theories, such as QED and QCD of the standard model[1]. Despite its great success, it has been known[1, 2] that the spinor matrix  $\gamma_5$  and chirality cannot in principle be well defined in the extended dimensions. Also it has trouble in applying directly to supersymmetric theories which depend dimension of space-time, and moreover it cannot keep track of the divergence behaviors (quadratic and above) of original integrals in the Feynman diagrams. So it is not useful in some calculations in effective field theories and chiral dynamics where we need to isolating the quadratic divergences for understanding the dynamical symmetry breaking and restoration.

Thus it is desirable to develop an alternative new regularization scheme which possesses the basic properties: being well defined in 4-dimensional space-time, preserving the gauge symmetry and Lorentz symmetry, keeping track of the divergent behaviors of original theories, making the practical calculations as simple as possible and applicable to both underlying and effective QFTs as well as supersymmetric and chiral QFTs.

Recently, a new regularization method proposed by one of us[3, 4] can satisfy all of the properties mentioned above and checked carefully with explicit calculations for many applications at one-loop level. For convenience, such new regularization is called the Loop Regularization since its prescription acts on the so-called irreducible loop integrals(ILIs)[3, 4]. For short, here we may use ‘LORE’ as an abbreviation of the loop regularization. It has been proved with explicit calculations at one loop level that the LORE method can preserve non-Abelian gauge symmetry [5] and supersymmetry [6]. It can provide a consistent calculation for the chiral anomaly[7] and the radiatively induced Lorentz and CPT-violating Chern-Simons term in QED[8] as well as the QED trace anomaly[9]. It allows us to derive the dynamically generated spontaneous chiral symmetry breaking of the low energy QCD for understanding the origin of dynamical quark masses and the mass spectra of light scalar and pseudoscalar mesons in a chiral effective field theory[10], and also to investigate the chiral symmetry restoration in a chiral thermodynamic model[11]. In particular, it enables us to consistently carry out the quantum gravitational contributions to gauge theories with asymptotic free power-law running[12–14].

It has been analyzed in ref.[3] that the LORE method can straightforwardly be generalized to higher loop calculations with an explicit demonstration on the general two loop integrals, i.e., the so-called  $\alpha\beta\gamma$  integrals. In fact, our general proof for the consistency of loop regularization via the  $\alpha\beta\gamma$  integrals was just following the same procedure which was adopted by ‘t Hooft and Veltman[1] to demonstrate the consistency of dimensional regularization. Since

the LORE method has been realized in four dimensional space-time without modifying the original theory, its consistency cannot be proved in the Lagrangian formalism to all orders, thus it is useful to develop a diagrammatic approach to make such a general proof. For that, we shall make explicit multiloop calculations to show its consistency, figuring out a general procedure for practical calculations, which is a further motivation in our present work. We are going to show in present paper that the evaluation of the irreducible loop integrals (ILIs) from Feynman integrals by adopting the ultraviolet divergence-preserving (UVDP) parametrization naturally leads to the Bjorken-Drell's circuit analogy between Feynman diagrams and electric circuits. As a consequence, when merging the LORE method with the Bjorken-Drell's circuit analogy, we arrive at the interesting observation that there is the one-to-one correspondence between the divergences of the UVDP parameters and the subdiagrams of Feynman diagrams, which enables us to extend the procedure to higher loop Feynman diagrams in a more general and systematic way.

The key concept in the LORE method is the introduction of the ILIs which are obtained from the Feynman diagrams by using the Feynman parametrization and the UVDP parametrization. A crucial point in the LORE method is the presence of two energy scales. They are introduced via the string-mode regulators in the regularization prescription acting on the ILIs. It has been shown that the two energy scales play the roles of the ultraviolet (UV) cut-off and infrared (IR) cut-off to avoid infinities without spoiling symmetries in the original theory[3, 4]. It is then inevitable to encounter the UV divergence in the UVDP parameter space due to the generic overlapping divergences. To be more explicit, we carry out a calculation for the general integrals, the so-called  $\alpha\beta\gamma$  integrals, arising from two loop Feynman diagrams, and show how to deal with the divergences in the UVDP parameter space by applying the LORE method. By identifying the divergences in the UVDP parameter space with those in the subdiagrams, we naturally arrive at the Bjorken-Drell's analogy between Feynman diagrams and electric circuits, where the UVDP parameters are found to be associated with the conductance or resistance in electric circuits. A detailed description on circuit analogy is given in the book by Bjorken and Drell[15] in which the circuit analogy was originally introduced to study the analyticity properties of Feynman diagrams from the causality requirement. In our present paper, we observe that the sets of conditions required to eliminate the overlapping divergent momentum integrals for evaluating the ILIs is analogous to the conservations of electric voltages in the loop, and the momentum conservations to the conservations of electric currents at each vertex. These equations are known as the Kirchhoff's laws in electric circuits. In particular, it is noticed that the divergence in Feynman diagrams corresponds to an infinite conductance or zero resistance in electric circuits. By adopting such an analogy, we perform a detailed calculation for one- and two-loop Feynman diagrams in the massive scalar  $\phi^4$  theory, and meanwhile we explicitly demonstrate the general procedure for applying the LORE method to multiloop calculations of Feynman diagrams.

We would like to emphasize that our motivation is not just for figuring out a much simpler regularization scheme, but for finding out whether there exists in principle a regularization scheme which can overcome some shortages and limitations in the widely-used regularization schemes. Meanwhile, we expect that such a regularization scheme must also be practical and as simple as possible. In fact, for the one loop calculation, the LORE method is really simple. For the higher-loop calculations, the procedure and calculation in the LORE method are not as concise as the ones in the dimensional regularization, but our treatment here makes the overlapping divergent structure as well as its divergent behavior physically manifest. To be more precise, the divergence structure for a diagram includes the overall quadratic or logarithmic divergence and the divergences in the subdiagrams, as well as corresponding

subtraction diagrams. Actually, the simplicity of the dimensional regularization is at the cost of three essential limitations: (i) the definition of  $\gamma_5$  in theories beyond 4 dimension, (ii) the requirement of exact dimension of original theories, (iii) preservation of quadratic divergence in original theories. To overcome such limitations in the dimensional regularization is our main purpose to look for a possible alternative consistent regularization scheme, which will be helpful for understanding deeply the applicability and consistency of QFTs. In this sense, the LORE method has been a step forward, as already shown in [5–14] at one loop level. It is worthwhile to go further and make an explicit check at higher loop level, which is our main goal in the present paper. Of course, for QFTs without three limitations in principle mentioned, the dimensional regularization scheme remains a powerful and simple one for a practical calculation.

The paper is organized as follows. In Sec. II, we briefly outline the LORE method and the concept of ILIs at one loop level. In Sec. III, a particular contribution of two-loop vacuum polarization diagram in QED is examined and show the general structure of overlapping divergences. It is then unavoidable to encounter the UV divergences hidden in the UVDP parameter space. In Sec. IV, we apply the LORE method to the general  $\alpha\beta\gamma$  integrals with  $\alpha = \gamma = 1$ ,  $\beta = 2$ , and explicitly show how the LORE method can appropriately regularize the UV divergences either from the original loop momenta or from the UVDP parameters. In Sec. V, we show how the evaluation of ILIs and UVDP parametrization naturally merges with the Bjorken-Drell's electric circuits analogy. In Sec. VI, The Bjorken-Drell's electric circuit analogy of Feynman diagrams allows us to analyze the origin of UV divergences contained in the UVDP parameter space, and to figure out the one-to-one correspondence of divergences between subdiagrams and UVDP parameters. In particular, the divergences in Feynman diagrams is shown to correspond to infinite conductances or zero resistances in electric circuits analogy. In Sec. VII, the LORE method combining with the Bjorken-Drell's analogy shows the advantage in analyzing a complicated overlapping divergence structure of Feynman diagrams. As an explicit illustration, the case with  $\alpha = \beta = \gamma = 1$  of the general  $\alpha\beta\gamma$  integral is discussed in detail and all the harmful divergences cancel exactly. As a practical application of all the machinery privously introduced, we carry out in Sec. VIII a detailed calculation of two loop contributions in the massive scalar  $\phi^4$  theory. Some additional quadratic corrections to the scalar mass are obtained, and leads to a power-law running. Based on the general analysis and explicit calculations, we arrive at in Sec. IX the general procedure of applying the LORE method to high-loop calculations. Our conclusions and remarks are presented in Sec. X.

## II. CONCEPT OF ILIS AND BRIEF OUTLINE ON THE LORE METHOD

We start from the fact that all Feynman integrals from the one-particle irreducible (1PI) graphs in 1-loop can be written, by using Feynman parametrization, in terms of the following sets of loop integrals,

$$I_{-2\alpha} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \mathcal{M}^2)^{2+\alpha}}, \quad \alpha = -1, 0, 1, 2, \dots,$$

for scalar type integrals and

$$\begin{aligned} I_{-2\alpha \ \mu\nu} &= \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - \mathcal{M}^2)^{3+\alpha}}, \\ I_{-2\alpha \ \mu\nu\rho\sigma} &= \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - \mathcal{M}^2)^{4+\alpha}}, \quad \alpha = -1, 0, 1, 2, \dots, \end{aligned} \quad (1)$$

for tensor type integrals, where the number  $(-2\alpha)$  in the subscript labels the dimension of power counting of energy momentum in the integrals. Thus two special cases  $\alpha = -1$  and  $\alpha = 0$  correspond to the quadratic divergent integrals ( $I_2, I_{2\mu\nu\dots}$ ) and the logarithmic divergent integrals ( $I_0, I_{0\mu\nu\dots}$ ). Note that the mass factor  $\mathcal{M}^2$  is in general a function of the Feynman parameters and the external momenta  $p_i$ , namely  $\mathcal{M}^2 = \mathcal{M}^2(m_1^2, p_1^2, \dots)$ .

The above loop integrals are the so-called one-fold irreducible loop integrals (ILIs)[3], which can be generalized to the n-fold ILIs evaluated from n-loop overlapping Feynman integrals of loop momenta  $k_i$  ( $i = 1, 2, \dots, n$ ). In general, the n-fold ILIs are defined as the loop integrals in which the overlapping momentum factor  $(k_i - k_j + p_{ij})^2$  ( $i \neq j$ ) originally appearing in the overlapping Feynman integrals has already been eliminated. It has been shown that any loop integrals can be evaluated into the corresponding ILIs by using both the Feynman parametrization and the UVDP parametrization methods[3]. Note that in the procedure of evaluating the ILIs, the algebraic computing for multi- $\gamma$  matrices involving loop momentum  $\not{k}$  such as  $\not{k}\gamma_\mu\not{k}$  should be carried out first and expressed in terms of the independent components:  $\gamma_\mu, \sigma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5$ .

The concept of ILIs is crucial in the LORE method. To see that, let us briefly examine the vacuum polarization in the non-abelian gauge theory. We begin with the following lagrangian in  $R_\xi$  gauge,

$$\mathcal{L} = \bar{\psi}_n(i\gamma^\mu D_\mu - m)\psi_n - \frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \partial^\mu \eta_a^* D_\mu \eta^a, \quad (2)$$

with

$$\begin{aligned} F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf_{abc}A_\mu^b A_\nu^c \\ D_\mu \psi_n &= (\partial_\mu + igT^a A_\mu^a)\psi_n, \end{aligned} \quad (3)$$

where  $\xi$  is a gauge parameter.  $\psi_n, A_\mu$  and  $\eta$  are fermions, gauge bosons and ghost fields, respectively.  $T^a$  are the generators of gauge group and  $f_{abc}$  the structure constants of the gauge group with  $[T^a, T^b] = if_{abc}T^c$ . The vacuum polarization corresponds to the self-energy diagrams of gauge boson, which contains the quadratically divergent integrals, the most divergent behavior in all of the Green functions in one-loop. Here we give the final results carried out by using the usual Feynman rules in the general  $\xi$  gauge. The details of the calculation can be found in ref.[3]. The explicit expressions for the gauge boson self-energy diagrams are given, in terms of the ILIs, as follows:

$$\begin{aligned} \Pi_{\mu\nu}^{(f)ab} &= -g^2 4N_f C_2 \delta_{ab} \int_0^1 dx [ (2a_2 - 1)I_2(m)g_{\mu\nu} \\ &\quad + 2x(1-x)(p^2 g_{\mu\nu} - p_\mu p_\nu)I_0(m) ], \end{aligned}$$

for the fermion loop contribution to the gauge self-energy diagram, and

$$\begin{aligned} \Pi_{\mu\nu}^{(g)ab} &= g^2 C_1 \delta_{ab} (p^2 g_{\mu\nu} - p_\mu p_\nu) \int_0^1 dx \{ [1 + 4x(1-x)] I_0 \\ &\quad + \frac{1}{2}\lambda [ (1 + 6x(1-x)(a_0 + 2) - 3a_0) I_0 - 2x(1-x)(1 + 12x(1-x)) p^2 I_{-2} ] \\ &\quad + \frac{3}{4}\lambda^2 a_{-2} x(1-x) p^2 I_{-2} \} \\ &\quad + g^2 C_1 \delta_{ab} \int_0^1 dx \{ 2(2a_2 - 1) I_2 g_{\mu\nu} + \lambda(a_0 - 1) p_\mu p_\nu x(1-x) p^2 I_{-2} \}, \end{aligned} \quad (4)$$

for the gauge boson and ghost loop contributions to the gauge self-energy diagram, where  $p$  is the momentum of the external gauge boson,  $N_f$  the number of fermions flavors,  $\lambda = 1 - \xi$ ,  $f_{acd}f_{bcd} = C_1\delta_{ab}$  and  $\text{tr}(T^a T^b) = C_2\delta_{ab}$ . We have also used the following definitions from the general Lorentz decomposition

$$I_{2\mu\nu} = a_2 I_2 g_{\mu\nu}, \quad I_{0\mu\nu} = \frac{1}{4}a_0 I_0 g_{\mu\nu}, \quad I_{-2\mu\nu} = \frac{1}{4}a_{-2} I_{-2} g_{\mu\nu}, \quad (5)$$

where  $I_{-2}$  and  $I_{-2\mu\nu}$  are convergent integrals with  $a_{-2} = 2/3$ . Note that  $\Pi_{\mu\nu}^{(g)ab}$  depends on the gauge parameter  $\xi$ . This is because the Green's functions are gauge dependent while only the S-matrix elements are gauge independent. However, current conservation implies that  $\Pi_{\mu\nu}^{(f)ab}$  and  $\Pi_{\mu\nu}^{(g)ab}$  have to satisfy the Ward Identities  $p^\mu \Pi_{\mu\nu}^{(f)ab} = 0$  and  $p^\mu \Pi_{\mu\nu}^{(g)ab} = 0$ . Notice that the first line of  $\Pi_{\mu\nu}^{(f)ab}$  and in the last line of  $\Pi_{\mu\nu}^{(g)ab}$  contain both quadratically and logarithmically divergent integrals which might violate gauge invariance. Only with the following consistency conditions

$$a_2 = 1/2, \quad a_0 = 1 \quad (6)$$

then the gauge invariance can be preserved.

Nevertheless, from the naive analysis of Lorentz decomposition and tensor manipulation, one gets by multiplying  $g^{\mu\nu}$  on both sides of Eq.(5),

$$\begin{aligned} g^{\mu\nu} I_{2\mu\nu} &= I_2 + \mathcal{M}^2 I_0 = 4a_2 I_2, \quad \text{i.e.} \quad a_2 = 1/4 + \mathcal{M}^2 I_0 / I_2, \\ g^{\mu\nu} I_{0\mu\nu} &= I_0 + \mathcal{M}^2 I_{-2} = a_0 I_0, \quad \text{i.e.} \quad a_0 = 1 + \mathcal{M}^2 I_{-2} / I_0, \end{aligned}$$

which leads, without using any regularization schemes, to the following relations

$$\begin{aligned} I_{2\mu\nu} &= \frac{1}{4}g_{\mu\nu} I_2 + \frac{1}{4}g_{\mu\nu}\mathcal{M}^2 I_0, \\ I_{0\mu\nu} &= \frac{1}{4}g_{\mu\nu} I_0 + g_{\mu\nu}\mathcal{M}^2 I_{-2} = \frac{1}{4}g_{\mu\nu} I_0 - \frac{i}{32\pi^2}g_{\mu\nu}. \end{aligned} \quad (7)$$

Clearly, the above naive relations for the divergent ILIs will destroy the gauge invariance. The reason is that in the divergent integrals which are generally not mathematically well defined without using proper regularization scheme, the tensor manipulation and integration do not commute with each other, so the result for divergent integration is not consistent in general. Thus in order to obtain a consistent result, one has to adopt a regularization scheme to make the divergent integrals well-defined. To see this, consider the time-time component on both sides of the relation for the quadratic divergent ILIs in Eq.(5)

$$I_{2\ 00} = a_2 I_2 g_{00}. \quad (8)$$

The Wick rotation will turn the four-dimensional energy momentum into Euclidean space and integrating over the zero component of energy momentum  $k_0$  on both sides, we get

$$\begin{aligned} I_2 &= -i \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \mathcal{M}^2} = -i \int \frac{d^3 k}{(2\pi)^4} \int dk_0 \frac{1}{k_0^2 + \mathbf{k}^2 + \mathcal{M}^2} \\ &= -i \int \frac{d^3 k}{(2\pi)^4} 2 \frac{1}{\sqrt{\mathbf{k}^2 + \mathcal{M}^2}} \tan^{-1} \left( k_0 / \sqrt{\mathbf{k}^2 + \mathcal{M}^2} \right) \Big|_{k_0=0}^{k_0=\infty} \\ &= -i \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{k}^2 + \mathcal{M}^2}} \end{aligned} \quad (9)$$

for the right-hand side, and

$$\begin{aligned}
I_{2\ 00} &= -i \int \frac{d^4 k}{(2\pi)^4} \frac{k_0^2}{(k^2 + \mathcal{M}^2)^2} = -i \int \frac{d^3 k}{(2\pi)^4} \int dk_0 \frac{k_0^2}{(k_0^2 + \mathbf{k}^2 + \mathcal{M}^2)^2} \\
&= -i \int \frac{d^3 k}{(2\pi)^4} \int dk_0 \left( \frac{1}{k_0^2 + \mathbf{k}^2 + \mathcal{M}^2} - \frac{\mathbf{k}^2 + \mathcal{M}^2}{(k_0^2 + \mathbf{k}^2 + \mathcal{M}^2)^2} \right) \\
&= -i \int \frac{d^3 k}{(2\pi)^4} \int dk_0 \left( \frac{1}{k_0^2 + \mathbf{k}^2 + \mathcal{M}^2} - \frac{1}{2} \frac{1}{k_0^2 + \mathbf{k}^2 + \mathcal{M}^2} \right) - \frac{k_0}{k_0^2 + \mathbf{k}^2 + \mathcal{M}^2} \Big|_{k_0=0}^{k_0=\infty} \\
&= \frac{-i}{2} \int \frac{d^3 k}{(2\pi)^4} 2 \frac{1}{\sqrt{\mathbf{k}^2 + \mathcal{M}^2}} \tan^{-1} \left( k_0 / \sqrt{\mathbf{k}^2 + \mathcal{M}^2} \right) \Big|_{k_0=0}^{k_0=\infty} \\
&= \frac{-i}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{k}^2 + \mathcal{M}^2}} = \frac{1}{2} I_2\ g_{00}
\end{aligned} \tag{10}$$

for the left-hand side. Note that the above integration over  $k_0$  is convergent, and should be safe for any algebraic manipulation. When comparing the results with both left and right hand sides, we obtain  $a_2 = 1/2$  which agrees with the consistency condition for gauge invariance. We then come to the conclusion that the general relation between the tensor-type and scalar-type quadratically divergent ILIs with  $a_2 = 1/2$  must be the exact consistency condition.

We would like to emphasize that the above demonstration for obtaining the consistency condition  $a_2 = 1/2$  between the quadratically divergent ILIs has nothing to do with any regularization schemes. Nevertheless, the drawback here is that it is obtained only for one of the Lorentz components rather than for the whole covariant Lorentz tensor. Thus it is necessary to look for a proper regularization scheme which can realize the consistency condition in a covariant way with the well-defined divergent integrals. Meanwhile, it should also preserve the original divergent behavior for both quadratical and logarithmic divergent integrals. Actually, it has explicitly been proved[3] that the LORE method does lead to the consistency conditions with  $a_2 = 1/2$  and  $a_0 = 1$ . A simple regularization prescription operating on the ILIs has been realized in four dimensional spacetime to satisfy the criteria mentioned in the introduction.

The regularization prescription of the LORE method is as follows: Firstly rotating the momentum to the four dimensional Euclidean space, then replacing the loop integrating variable  $k^2$  and the loop integrating measure  $\int d^4 k$  of the ILIs by the corresponding regularized ones  $[k^2]_l$  and  $\int [d^4 k]_l$ :

$$\begin{aligned}
k^2 &\rightarrow [k^2]_l \equiv k^2 + M_l^2, \\
\int d^4 k &\rightarrow \int [d^4 k]_l \equiv \lim_{N, M_l^2} \sum_{l=0}^N c_l^N \int d^4 k,
\end{aligned} \tag{11}$$

where  $M_l^2$  ( $l = 0, 1, \dots$ ) may be regarded as the regulator masses for the ILIs. The coefficients  $c_l^N$  and the regulator masses are chosen to satisfy the following conditions:

$$\lim_{N, M_l^2} \sum_{l=0}^N c_l^N (M_l^2)^n = 0 \quad (n = 0, 1, \dots), \tag{12}$$

where the notation  $\lim_{N, M_l^2}$  denotes the limiting case  $\lim_{N, M_R^2 \rightarrow \infty}$ . The initial conditions  $M_0^2 = 0$  and  $c_0^N = 1$  are taken to recover the original integrals in the limit  $M_l^2 \rightarrow \infty$  ( $l = 1, 2, \dots$ ).

With the above regularization prescription, we have shown that the regularized 1-fold ILIs satisfy the following consistency conditions[3]:

$$\begin{aligned} I_{2\mu\nu}^R &= \frac{1}{2} g_{\mu\nu} I_2^R, & I_{2\mu\nu\rho\sigma}^R &= \frac{1}{8} (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\rho\nu}) I_2^R, \\ I_{0\mu\nu}^R &= \frac{1}{4} g_{\mu\nu} I_0^R, & I_{0\mu\nu\rho\sigma}^R &= \frac{1}{24} (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\rho\nu}) I_0^R. \end{aligned} \quad (13)$$

which are actually the necessary and sufficient conditions to preserve the gauge symmetry in QFTs. Here the superscript “R” denotes the regularized ILIs. Note that the dimensional regularization scheme also leads to  $a_2 = 1/2$  for  $\mathcal{M}^2 \neq 0$  and  $a_0 = 1$ , while the resulting  $I_2^R$  is suppressed to be a logarithmic divergence multiplying by the mass scale  $\mathcal{M}^2$ , thus it goes to vanish  $I_2^R = 0$  when  $\mathcal{M}^2 = 0$ . This is the well-known fact that the dimensional regularization does not preserve the quadratic divergent behavior of the original loop integrals.

As the simplest solution of Eq. (12), take the string-mode regulators

$$M_l^2 = \mu_s^2 + l M_R^2, \quad (14)$$

with  $l = 1, 2, \dots$ , then the coefficients  $c_l^N$  are completely determined to be

$$c_l^N = (-1)^l \frac{N!}{(N-l)!!!}, \quad (15)$$

where  $M_R$  may be regarded as a basic mass scale of loop regulator. When applying the above prescription and solution to the ILIs, the regularized ILIs in the Euclidean space-time are generally expressed as follows:

$$\begin{aligned} I_{-2\alpha}^R &= i(-1)^\alpha \lim_{N, M_l^2} \sum_{l=0}^N c_l^N \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + M^2 + M_l^2)^{2+\alpha}}, \\ I_{-2\alpha \mu\nu}^R &= -i(-1)^\alpha \lim_{N, M_l^2} \sum_{l=0}^N c_l^N \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 + M^2 + M_l^2)^{3+\alpha}}, \quad \alpha = -1, 0, 1, 2, \dots \\ I_{-2\alpha \mu\nu\rho\sigma}^R &= i(-1)^\alpha \lim_{N, M_l^2} \sum_{l=0}^N c_l^N \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 + M^2 + M_l^2)^{4+\alpha}} \end{aligned} \quad (16)$$

For the regularized quadratically and logarithmically divergent ILIs  $I_2^R$  and  $I_0^R$ , we have shown that they have the following explicit experssions[3]:

$$\begin{aligned} I_2^R &= \frac{-i}{16\pi^2} \left\{ M_c^2 - \mu^2 \left[ \ln \frac{M_c^2}{\mu^2} - \gamma_w + 1 + y_2 \left( \frac{\mu^2}{M_c^2} \right) \right] \right\} \\ I_0^R &= \frac{i}{16\pi^2} \left[ \ln \frac{M_c^2}{\mu^2} - \gamma_w + y_0 \left( \frac{\mu^2}{M_c^2} \right) \right] \end{aligned} \quad (17)$$

with  $\mu^2 = \mu_s^2 + M^2$ , and

$$\begin{aligned} \gamma_w &\equiv \lim_N \left\{ \sum_{l=1}^N c_l^N \ln l + \ln \left[ \sum_{l=1}^N c_l^N l \ln l \right] \right\} = \gamma_E = 0.5772 \dots, \\ y_0(x) &= \int_0^x d\sigma \frac{1 - e^{-\sigma}}{\sigma}, \quad y_1(x) = \frac{e^{-x} - 1 + x}{x} \\ y_2(x) &= y_0(x) - y_1(x), \quad \lim_{x \rightarrow 0} y_i(x) \rightarrow 0, \quad i = 0, 1, 2 \\ M_c^2 &\equiv \lim_{N, M_R} M_R^2 \sum_{l=1}^N c_l^N (l \ln l) = \lim_{N, M_R} M_R^2 / \ln N \end{aligned} \quad (18)$$



which indicates that the  $\mu_s$  sets an IR ‘cutoff’ at  $M^2 = 0$  and  $M_c$  provides an UV ‘cutoff’. For renormalizable QFTs,  $M_c$  can be taken to be infinity ( $M_c \rightarrow \infty$ ). In a theory without infrared divergence,  $\mu_s$  can safely be taken as  $\mu_s = 0$ . In fact, by taking  $M_c \rightarrow \infty$  and  $\mu_s = 0$ , we recover the initial integral of the theory. Also by taking  $M_R$  and  $N$  to infinity, we arrive at a regularized theory which becomes independent of the regularization prescription. Note that the function  $y_0(x)$  with  $x = \mu^2/M_c^2$  is actually the incomplete gamma function, which has the property:  $y_0(x) \rightarrow 0$  at  $x \rightarrow 0$  (i.e., in the limit  $M_c \rightarrow \infty$ ). In comparison with the dimensional regularization, there is a correspondence:  $\ln \frac{M_c^2}{\mu^2} \rightarrow \frac{2}{\varepsilon}$  with  $M_c \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , which indicates that the function  $y_0(x)$  approaches to zero much faster than the polynomial of  $\varepsilon$  in the dimensional regularization. This can be seen explicitly from the expression:  $y_0(x) \simeq x \sim e^{-\frac{2}{\varepsilon}} \rightarrow 0$  in the limit  $M_c \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ .

We would like to point out that the prescription in the LORE method looks very similar to the Pauli-Villars prescription. Nevertheless, the basic concept is quite different as the prescription in the LORE method is acting on the ILIs rather than on the propagators in the Pauli-Villars scheme. This is why the LORE method can preserve non-Abelian gauge symmetry, while the Pauli-Villars regularization can not. In this sense, we would like to emphasize that the concept of ILIs is a crucial point in the LORE method to realize the interesting symmetry-preserving regularization scheme. In particular, the introduction of two intrinsic energy scales without spoiling symmetries of original theory is an advantage in the LORE method to avoid the infinities of divergent Feynman integrals [3–14]. For the effective theories, the intrinsic UV ‘cutoff’ scale  $M_c$  plays the role as the characteristic energy scale below which the physics can be well described by the effective quantum field theory[10–14].

### III. OVERLAPPING DIVERGENCES AND UVDP PARAMETRIZATION

It is well-known that for Feynman diagrams beyond one-loop order, a new feature, *overlapping* divergences. It happens when two divergent loops share a common propagator. To illustrate this, consider one particular contribution to the photon vacuum polarization at two-loop order of quantum electrodynamics (QED) (see Fig. 1)

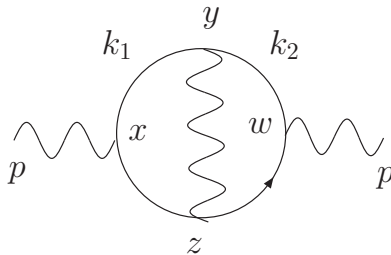
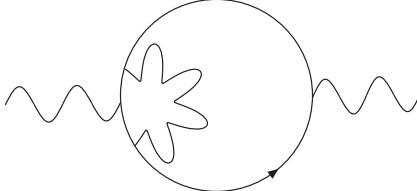


FIG. 1:

Here we may follow the argument in the textbook[16]. As discussed in the usual texts of QFTs, the UV divergences in Fig. 1 can arise from several regions of momentum spaces. One divergent contribution comes from the region where there is a large momentum passing through the left subdiagram. This means that the three points  $x, y$ , and  $z$  in position space are very close together, while the point  $w$  is farther away. In this region one can think that the virtual photon gives the corrections to the vertex  $x$ . Plug the divergent part of the

one-loop vertex corrections into the rest of diagram and integrate over  $k_1$ . We get expression identical to the one-loop photon vacuum polarization correction multiplied by the additional logarithmic divergence, as shown in Fig. 2. Obviously, a similar divergent contribution to the diagram in Fig.1 arises from the region with a large momentum passing through the right subdiagram as shown in Fig. 2. It is manifest that the  $\log^2 \Lambda^2$  term comes from the



$$\sim \alpha(g^{\mu\nu}p^2 - p^\mu p^\nu)(\log \Lambda^2 - \log p^2) \cdot \alpha \log \Lambda^2$$

FIG. 2:

region where both  $k_1$  and  $k_2$  are large. While the  $\log p^2 \log \Lambda^2$  term results from the region where  $k_2$  is large but  $k_1$  is small, another such a term arises from the region where  $k_1$  is large but  $k_2$  is small. The terms like  $\log p^2 \log \Lambda^2$  are called *nonlocal* or *harmful* divergences as such terms cannot be canceled by the ordinary subtraction scheme by introducing the corresponding two loop counterterms in the Lagrangian.

It is then expected that these harmful divergences are canceled by two types of counterterm diagrams. First, we can build diagrams of order  $\alpha^2$  by inserting the order- $\alpha$  counterterm vertex into the one-loop vacuum polarization diagram (see Fig. 3). Such two diagrams

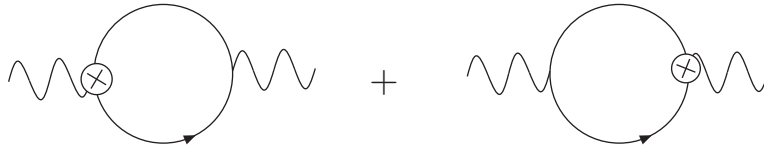


FIG. 3:

should cancel the harmful divergences as shown in Fig.2. Once these counterterm diagrams are added, the only divergence left is exactly local and can be canceled by the two-loop overall counterterm, which is diagrammatically represented in Fig. 4.



FIG. 4: local counter term

The discussion given above is a general description in the ordinary textbooks and there is no problem in principle. However, in the practical calculation, one actually meets some conceptual problems. There is no doubt that we have to integrate over two loop momenta  $k_1$  and  $k_2$  one by one. Suppose that we first integrate over the loop momentum  $k_1$ , which

means that we integrate over the left subdiagram corresponding to the left vertex insertion. Then we integrate over the loop momentum  $k_2$ , which corresponds to the overall divergence of the whole diagram as its divergent behavior is easily found to be quadratic from the simple power counting. But the problem arises when we look into which loop momentum integral represents the right subdiagram corresponding correction to the right vertex. This is because we have already integrated over both loop momenta in the diagram with the above procedure. It appears that we have nothing to do with it. Actually, when carrying out the calculations by using the Feynman parametrization and UVDP parametrization to combine the momenta in the denominator, we will find that, besides of the divergences coming from the integral of the two loop momenta  $k_1$  and  $k_2$ , the integrations over the UVDP parameters are also logarithmically divergent, which exactly reproduce the divergence behavior of the vertex correction at one-loop order. This observation makes it clear that the integration of right subdiagram is “hidden” in or transformed into the parameter space with the usual procedure of dealing with the two-loop overlapping diagrams.

Thus the next immediate question is whether, given a divergence in the UVDP parameter space, we can find out the origin of this divergence in the original Feynman diagrams. Our answer is positive. This is actually the main purpose in our present paper. We shall show that there is an exact correspondence between the UVDP parameter integrals and those from the original loop momenta. The key conceptual tool for arriving at this conclusion is the observation of the Bjorken-Drell’s analogy between the Feynman diagrams and electrical circuits, which will be demonstrated below.

Before proceeding, it is interesting to note that all the overlapping divergent integrals (including scalar-type and tensor-type) of two-loop Feynman diagrams in QED can be reduced to the following two types of integrals by adopting the Feynman parametrization:

$$I_{111} = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{(k_1^2 - m_1^2)(k_2^2 - m_2^2)[(k_1 - k_2 + p)^2 - m_3^2]}, \quad (19)$$

$$I_{121} = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{(k_1^2 - m_1^2)(k_2^2 - m_2^2)^2[(k_1 - k_2 + p)^2 - m_3^2]}, \quad (20)$$

where  $m_i^2$  are in general the functions of the external momenta  $p$  and Feynman parameters. Such integrals are actually the two special cases of the general  $\alpha\beta\gamma$  integrals[1]. Therefore, it is useful to make a general discussion and analysis on the regularization and renormalization for the general  $\alpha\beta\gamma$  integrals.

In order to avoid the complication involving the reducible loop integrals and tensor-type integrals, we may consider only scalar-type ILIs. As pointed out in Ref. [1] by ’t Hooft and Veltman, a general two-loop order Feynman diagram can be reduced to the general  $\alpha\beta\gamma$  integrals of the form:

$$I_{\alpha\beta\gamma} = \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{(k_1^2 - m_1^2)^\alpha (k_2^2 - m_2^2)^\beta [(-k_1 - k_2 + p)^2 - m_3^2]^\gamma}. \quad (21)$$

We shall focus on the problem how to disentangle the overlapping divergences with the LORE method. Especially, we will show how to deal with the divergences contained in the UVDP parameter space caused by the overlapping structure. With the advantage of the UVDP parametrization in evaluating the ILIs from Feynman diagrams, we naturally arrive at the Bjorken-Drell’s analogy between Feynman diagrams and electrical circuit diagrams. This powerful tool gives us exact one-to-one correspondence of the divergences between the parameter space and the subdiagrams, which can explicitly be demonstrated. As a consequence, it straightforwardly leads to the important theorem on the cancelation of

harmful divergences. To realize those goals, it is enough to keep track of only the overlapping divergences, such as the terms  $M_c^2 \cdot \log \frac{M_c^2}{-p^2}$  and  $\log \frac{M_c^2}{-p^2} \cdot \log \frac{M_c^2}{-p^2}$ . For the harmless divergences and finite terms, they can be either absorbed into the two-loop overall counterterms or kept in the final expression.

From the general form of Eq.(21), one can easily recognize that there are in general one overall integral  $\alpha\beta\gamma$  and three subintegrals ( $\alpha\beta$ ,  $\beta\gamma$  and  $\gamma\alpha$ ), represented diagrammatically as the following three corresponding subdiagrams (See Fig.(5) and Fig.(6)). The corresponding counterterm diagrams are shown in Fig.(7), which are generally needed for the cancelation of the harmful divergences.

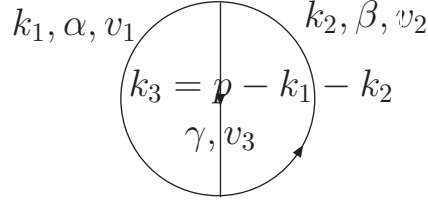


FIG. 5:  $\alpha\beta\gamma$  diagram

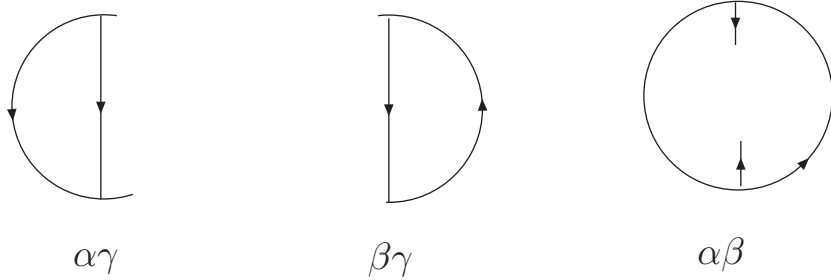


FIG. 6: subdivergences

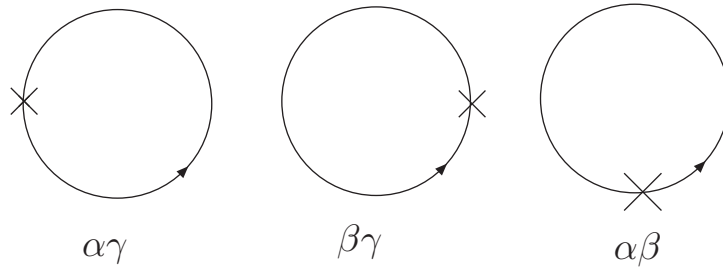


FIG. 7: counterterm diagrams

By power counting, it is easy to see from Eq.(21) that there are only two cases which involve the overlapping divergences, i.e., (1)  $\alpha + \beta + \gamma = 4$ , and (2)  $\alpha + \beta + \gamma = 3$ . Other cases

with  $\alpha + \beta + \gamma > 4$  contain only harmless divergences and the overall integral is convergent. Thus we shall only discuss these two cases.

As the first step, we shall write the general  $\alpha\beta\gamma$  integral given in Eq.(21) into the ILIs. With the standard manipulations, such as combining factors in the denominator with the UVDP parametrization and making translation of loop momenta, we can then get rid of the cross terms of momenta in the denominator. Some useful formula and further discussions on the UVDP parametrization method are given in Appendix B, which enables us to reexpress Eq.(21) into the following form

$$\begin{aligned}
I_{\alpha\beta\gamma} &= \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^2} \delta(1 - \sum_{i=1}^3 \frac{1}{1+v_i}) \\
&\quad \frac{\frac{1}{(1+v_1)^{\alpha-1}} \frac{1}{(1+v_2)^{\beta-1}} \frac{1}{(1+v_3)^{\gamma-1}}}{\{\frac{1}{1+v_1}(k_1^2 - m_1^2) + \frac{1}{1+v_2}(k_2^2 - m_2^2) + \frac{1}{1+v_3}[(-k_1 - k_2 + p)^2 - m_3^2]\}^{\alpha+\beta+\gamma}} \\
&= \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^{1+\alpha_i}} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \int \frac{d^4 k'_1}{(2\pi)^4} \int \frac{d^4 k'_2}{(2\pi)^4} \\
&\quad \frac{1}{[(\frac{1}{1+v_1} + \frac{1}{1+v_3})k'_1{}^2 + \frac{3+v_1+v_2+v_3}{(2+v_1+v_2)(1+v_3)}k'_2{}^2 + \frac{1}{3+v_1+v_2+v_3}p^2 - \sum_{j=1}^3 \frac{m_j^2}{1+v_j}]^{\alpha+\beta+\gamma}},
\end{aligned} \tag{22}$$

where we have used  $\alpha_i$  ( $i=1,2,3$ ) to denote  $\alpha, \beta, \gamma$  respectively, and made the following momentum translation

$$k'_1 = k_1 + \frac{1+v_1}{2+v_1+v_3}(p - k_2), \tag{23}$$

$$k'_2 = k_2 + \frac{1+v_2}{3+v_1+v_2+v_3}p. \tag{24}$$

Below we shall drop the prime on  $k'_i$  for simplicity. It is seen that the cross term of momentum is eliminated.

In general, the power indices  $\alpha, \beta, \gamma$  are positive integers, so that we have  $\alpha + \beta + \gamma \geq 3$ . Thus the above integral is convergent with respect to one of loop momentum  $k_i$ s. From the general structure of Eq.(21), it is clear that the final result is independent of the integration order over  $k_1$  and  $k_2$ . Without loss of generality, we can first integrate over  $k_1$  and explicitly obtain the expression:

$$\begin{aligned}
I_{\alpha\beta\gamma} &= \frac{i}{16\pi^2} \frac{\Gamma(\alpha + \beta + \gamma - 2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 dv_i \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) F(v_k) \\
&\quad \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{[k_2^2 - \mathcal{M}^2(p^2, m_k^2, v_k)]^{\alpha+\beta+\gamma-2}},
\end{aligned} \tag{25}$$

with

$$\mathcal{M}^2 = -\frac{(2+v_1+v_3)(1+v_2)}{(3+v_1+v_2+v_3)^2} p^2 + \frac{(2+v_1+v_3)(1+v_2)}{3+v_1+v_2+v_3} \sum_{j=1}^3 \frac{m_j^2}{1+v_j}, \tag{26}$$

$$F(v_k) = \frac{(2+v_1+v_3)^{\alpha+\beta+\gamma-4} (1+v_1)^{1-\alpha} (1+v_3)^{1-\gamma} (1+v_2)^{\alpha+\gamma-3}}{(3+v_1+v_2+v_3)^{\alpha+\beta+\gamma-2}}. \tag{27}$$

The above result is symmetric under the interchange between  $v_1$  ( $m_1, \alpha_1 = \alpha$ ) and  $v_3$  ( $m_3, \alpha_3 = \gamma$ ). In fact, the original expression is also symmetric under the permutations among  $v_1$  ( $m_1, \alpha_1 = \alpha$ ),  $v_2$  ( $m_2, \alpha_2 = \beta$ ) and  $v_3$  ( $m_3, \alpha_3 = \gamma$ ). Making the following scaling transformation for the momentum

$$k_2^2 = \frac{(2 + v_1 + v_3)(1 + v_2)}{3 + v_1 + v_2 + v_3} l_+^2, \quad (28)$$

we then obtain the following more symmetric expression

$$I_{\alpha\beta\gamma} = \frac{i}{16\pi^2} \frac{\Gamma(\alpha + \beta + \gamma - 2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1 + v_i)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1 + v_j}) F(v_k) \int \frac{d^4 l_+}{(2\pi)^4} \frac{1}{[l_+^2 - \mathcal{M}^2(p^2, m_k^2, v_k)]^{\alpha+\beta+\gamma-2}}, \quad (29)$$

with

$$F(v_k) = \frac{(1 + v_1)^{3-\alpha}(1 + v_2)^{3-\beta}(1 + v_3)^{3-\gamma}}{(3 + v_1 + v_2 + v_3)^2},$$

$$\mathcal{M}^2 = \sum_{j=1}^3 \left( \frac{m_j^2}{1 + v_j} \right) - \frac{1}{3 + v_1 + v_2 + v_3} p^2.$$

In the subsequent sections, we will show how this formula can naturally be obtained when merging the UVDP parametrization and the evaluation of ILIs with the Bjorken-Drell's circuit analogy.

To go further, we need to consider some explicit values of  $\alpha, \beta, \gamma$ . As mentioned above, the only cases involving overlapping divergences are (1)  $\alpha + \beta + \gamma = 4$  and (2)  $\alpha + \beta + \gamma = 3$ . Up to the field redefinition, we can always take the corresponding cases to be (1)  $\alpha = \gamma = 1, \beta = 2$  and (2)  $\alpha = \beta = \gamma = 1$ . We will consider these two cases separately in detail.

#### IV. TREATMENT OF DIVERGENCES IN THE UVDP PARAMETER SPACE

Let us first consider the simpler case with  $\alpha = \gamma = 1, \beta = 2$ , where the general form of  $\alpha\beta\gamma$  integral Eq.(25) can be simplified into the following form

$$I_{121} = \frac{i}{16\pi^2} \int_0^\infty \prod_{i=1}^3 dv_i \delta(1 - \sum_{j=1}^3 \frac{1}{1 + v_j}) \frac{1}{(3 + v_1 + v_2 + v_3)^2(1 + v_2)} \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{[k_2^2 - \mathcal{M}(p^2, m_k^2, v_k)]^2} \rightarrow -\frac{1}{(16\pi^2)^2} \int_0^\infty \prod_{i=1}^3 dv_i \delta(1 - \sum_{j=1}^3 \frac{1}{1 + v_j}) \frac{1}{(3 + v_1 + v_2 + v_3)^2(1 + v_2)} (\ln \frac{M_c^2}{\mathcal{M}_s^2} - \gamma_\omega + y_0(\frac{\mathcal{M}_s^2}{M_c^2})). \quad (30)$$

The integral over the loop momentum  $k_2$  above is logarithmically divergent, which represents the overall divergence. When carrying out the integration over the loop momentum  $k_2$ , we have applied the LORE method to regularize it, where  $\mathcal{M}_s^2 = \mathcal{M}^2 + \mu_s^2$  with  $\mu_s^2$  playing

the role of IR divergence cut-off. In the following, we are always working in the massive theory, so there is no IR problem and the scale  $\mu_s$  can safely be set to  $\mu_s^2 = 0$ . Also, in the limit  $M_c^2 \rightarrow \infty$ ,  $y_i(\frac{M_s^2}{M_c^2}) \simeq O(\frac{M_s^2}{M_c^2}) \rightarrow 0$ , so  $y_i$ 's vanish identically. By power counting, the only contribution from  $y_i$ 's may arise when the overall quadratic divergence is multiplied by  $y_0(\frac{M_s^2}{M_c^2})$ . Such a contribution is finite and does not disturb the divergent terms. Nevertheless, in the present paper, we only focus on the divergent part to show the consistency of the LORE method. Thus, the limit  $M_c^2 \rightarrow \infty$  is always taken and all the terms  $y_i(\frac{M_s^2}{M_c^2})$  are dropped below to simplify our expressions.

It is not difficult to see that there exists a divergence in the region of UVDP parameter space with  $v_1, v_3 \rightarrow \infty$ , which reflects the divergence of subdiagram  $\alpha\gamma$ . To extract the divergence, we may focus on the region where  $v_1, v_3 > V$  with  $V \gg 1$ . In such a region,  $v_2 \rightarrow 0$ , which is ensured by the delta function, so the domain of the integration is transformed into  $\int_V^\infty dv_1 \int_V^\infty dv_3 \int_0^\infty dv_2$ . With such a treatment, it is easy to check that  $\mathcal{M} \rightarrow m_2^2$  and some terms small in comparison with  $v_1$  and  $v_3$  can be neglected. Thus the integral  $I_{121}$  is simplified into the following form

$$\begin{aligned} I_{121} &\simeq -\frac{1}{(16\pi^2)^2} \int_V^\infty dv_1 \int_V^\infty dv_3 \int_0^\infty dv_2 \delta(1 - \frac{1}{1+v_2}) \\ &\quad \frac{1}{(v_1 + v_3)^2} (\ln \frac{M_c^2}{m_2^2} - \gamma_\omega) \\ &= -\frac{1}{(16\pi^2)^2} (\ln \frac{M_c^2}{m_2^2} - \gamma_\omega) \int_V^\infty dv_1 \frac{1}{v_1 + V}, \end{aligned} \quad (31)$$

where we integrate over  $v_2$  and  $v_3$  as they are convergent. The remaining integration over  $v_1$  is divergent, which has to be regularized appropriately. The LORE method has been shown to be more suitable in this situation[3, 4], because such a divergence is a kind of scalar type divergent ILIs, which is the object that can be regularized in the LORE method, rather than other physical objects, such as propagators or the dimension of the theory. To regularize the UVDP parameter integral, it is more useful to transform it into a manifest ILI. For that, one just needs to multiply a free mass-squared scale  $q_o^2$  to  $v_1$ , which will be determined by a suitable criterion. Eventually, it will cause the harmful divergences of different diagrams to be canceled. In general, such a scale can be the function of the intrinsic quantities in the theory, such as masses of particles or external momenta.

$$\begin{aligned} \int_V^\infty d(q_o^2 v_1) \frac{1}{q_o^2 v_1 + q_o^2 V} &= \int_{(q_o^2)V}^\infty dq_1^2 \frac{1}{q_1^2 + q_o^2 V} \\ &= \ln \frac{M_c^2}{2q_o^2 V} - \gamma_\omega, \end{aligned} \quad (32)$$

where we have defined  $q_1^2 \equiv q_o^2 v_1$ . In the following, we will frequently encounter similar divergent integrals in the UVDP parameter space. Unless specified explicitly, we shall always use this prescription to deal with them. Here we would like to emphasize that the above prescription is the only one consistent within the framework of the LORE method. For other regularization schemes, such an approach cannot give consistent results, such as the Pauli-Villars regularization in which the regularized objects are the propagators of internal particles.

It can be shown that in other regions of parameter space, there are no further divergences. Namely, besides the overlapping divergence given above, they contain only the harmless

overall divergence from the integration of  $k_2$ . Thus the general form of overlapping divergence in the integral  $I_{121}$  can be written as:

$$I_{121} \simeq -\frac{1}{(16\pi^2)^2} (\ln \frac{M_c^2}{m_2^2} - \gamma_\omega) \cdot (\ln \frac{M_c^2}{2q_o^2 V} - \gamma_\omega) \quad (33)$$

In order to show the exact cancelation of harmful divergences for  $I_{121}$ , it is necessary to calculate its corresponding counterterm diagram ( $\alpha\gamma$ ). (see Fig. (7))

$$I_{121}^{(c)(\alpha\gamma)} = - \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{(k_2^2 - m_2^2)^2} \text{DP} \left\{ \int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{(k_1^2 - m_1^2)[(k_1 - k_2 + p)^2 - m_3^2]} \right\} \quad (34)$$

where  $\text{DP}\{\}$  denotes the divergent part. Such a counterterm integral can be easily computed,

$$I_{121}^{(c)(\alpha\gamma)} = +\frac{1}{(16\pi^2)^2} (\ln \frac{M_c^2}{m_2^2} - \gamma_\omega) \cdot (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) \quad (35)$$

where the second factor comes from the subintegral ( $\alpha\gamma$ ) part contained in  $\text{DP}\{\}$  and the first one from the integration of internal loop momentum  $k_2$ .

It is obvious that there is an exact correspondence between the factors in each expression. When taking the free scale to be  $\mu^2 = 2q_o^2 V$ , the two divergent terms cancel each other exactly. Here the divergence contained in the UVDP parameter space in the region  $v_1, v_3 \rightarrow \infty$  reproduces that of subintegral ( $\alpha\gamma$ ), namely the integration over  $k_1$ . We also notice that the divergences of  $I_{121}$  are factorizable and can be written as the product of two divergent integrals, i.e., one from the integral  $k_2$  for the overall divergence and the other from the subintegral  $k_1$  ( $\alpha\gamma$ ) for the sub-divergence, which is transformed into and represented in the UVDP parameter integral of the region  $v_1, v_3 \rightarrow \infty$ . This is the general feature when using the LORE method to disentangle the two-loop overlapping divergences. We will make a more explicit demonstration on this feature below by merging with the Bjorken-Drell's analogy between Feynman diagrams and electrical circuits.

## V. EVALUATION OF ILIS AND BJORKEN-DRELL'S ANALOGY BETWEEN FEYNMAN DIAGRAMS AND ELECTRICAL CIRCUIT DIAGRAMS

In order to generalize the correspondence between the divergences in the UVDP parameter space and those in the subintegrals to more complicated cases, it is interesting to observe that the UVDP parametrization and the evaluation of ILIs in the LORE method naturally merge with the Bjorken-Drell's analogy between the general Feynman diagrams and the electrical circuits. A detailed description for such the analogy is referred to the book by Bjorken and Drell[15]. It was originally motivated for discussing the analyticity properties of Feynman diagrams from the causality requirement. Here let us first establish such the analogy by developing a standard procedure and notation following Bjorken and Drell, and then apply it to the general  $\alpha\beta\gamma$  integrals by merging it to the LORE method.

For a general connected Feynman diagram, we shall always denote the external momenta of the diagram by  $p_1, \dots, p_m$  with the direction of coming into the diagram. Thus, according to overall momentum conservation, we have:

$$\sum_{s=1}^m p_s = 0. \quad (36)$$



To each internal line we assign a momentum  $k_j$  with a specified direction and a mass  $m_j$ . At each vertex, we have a law of momentum conservation of the form

$$\sum_{j=1}^n \epsilon_{ij} k_j + \sum_{s=1}^m \bar{\epsilon}_{is} p_s = 0, \quad (37)$$

where  $\epsilon_{ij}$  is chosen to be +1 if internal line  $j$  enters vertex  $i$ , while  $-1$  if internal line  $j$  leaves vertex  $i$ , otherwise  $\epsilon_{ij}$  is defined to be 0.  $\bar{\epsilon}_{is}$  has the similar definition for the external lines which, by convention, are always taken to enter vertices.

Each diagram has a definite number  $k$  of internal loops. However, we have the freedom to choose the concrete internal loops and assign each loop a momentum  $l_r$  which are going to be integrated out along the loop. Thus, for each internal line  $j$ , we have the following decomposition:

$$k_j = q_j + \sum_{r=1}^k \eta_{jr} l_r, \quad (38)$$

where  $\eta_{jr}$  is chosen to be 1 if the  $j$ th internal line lies on the  $r$ th loop and the momenta  $k_j$  and  $l_r$  are parallel, and  $-1$  if the  $j$ th line lies on the  $r$ th loop but  $k_j$  and  $l_r$  are antiparallel, otherwise  $\eta_{jr}$  is 0. Notice that here we introduce another kind of internal momentum  $q_j$ , which will be determined after we adopt the UVDP parametrization for combining denominators to evaluate the ILIs. From the decomposition Eq.(38), we can immediately obtain the following momentum conservation law for each vertex in terms of  $q_j$ :

$$\sum_{j=1}^n \epsilon_{ij} q_j + \sum_{s=1}^m \bar{\epsilon}_{is} p_s = 0, \quad (39)$$

which follows from Eq.(37) and

$$\sum_{j=1}^n \epsilon_{ij} \eta_{jr} = 0, \quad (40)$$

which is a consequence of the definitions of  $\epsilon_{ij}$  and  $\eta_{jr}$  given in Eqs. (37) and (38).

The general structure of the Feynman integral can be written as follows:

$$I(p_1, \dots, p_m) = \int d^4 l_1 \dots d^4 l_k \frac{N}{(k_1^2 - m_1^2)^{\alpha_1} \dots (k_n^2 - m_n^2)^{\alpha_n}}, \quad (41)$$

where  $N$  represents the numerator of a general matrix element, which can be the products of external momenta, internal momenta, spin matrices, wave functions and so on. By adopting the UVDP parametrization, the above integral can be written as:

$$\begin{aligned} I(p_1, \dots, p_m) &= \int d^4 l_1 \dots d^4 l_k \frac{\Gamma(\sum_{j=1}^n \alpha_j)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^n \frac{1}{1+v_j}) \\ &\quad \frac{N}{[\sum_{j=1}^n \frac{k_j^2 - m_j^2}{1+v_j}]^{\sum_{j=1}^n \alpha_j}} \\ &= \frac{\Gamma(\sum_{j=1}^n \alpha_j)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int d^4 l_1 \dots d^4 l_k \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^n \frac{1}{1+v_j}) \\ &\quad \frac{N}{[\sum_{j=1}^n \frac{q_j^2 - m_j^2}{1+v_j} + 2 \sum_{j,r} \frac{q_j \eta_{jr} l_r}{1+v_j} + \sum_{j,r,r'} \frac{\eta_{jr} \eta_{jr'} l_r l_{r'}}{1+v_j}]^{\sum_{j=1}^n \alpha_j}}. \end{aligned} \quad (42)$$

In order to obtain the required ILIs, we need to eliminate the cross terms in the denominator which implies

$$\sum_{j=1}^n \frac{\eta_{jr} q_j}{1 + v_j} = 0 \quad (43)$$

for each loop  $r = 1, \dots, k$ . Now we have the enough conditions Eqs. (39) and (43) to determine the momenta  $q_j$  for each diagram. The above procedure is essentially equivalent to the usual way of shifting the loop momenta for completing the square in the denominator. Our next task is to diagonalize the momentum integration variables so that we can integrate over each momentum integrals separately in Eq. (42).

Before doing the calculation, let us try to understand Eqs.(39) and (43) from an alternative interesting perspective. First we put them into a more heuristic form:

$$\sum_{q_j \text{ in loop } r} \frac{q_j}{1 + v_j} = 0, \quad (44)$$

$$\sum_{q_j, p_s \text{ entering vertex } i} (q_j + p_s) = 0, \quad (45)$$

we then arrive at a complete analogy between the Feynman diagrams and electrical circuits.

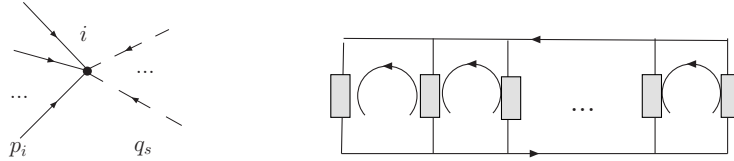


FIG. 8: Left: Current conservation for each vertex  $i$ ; Right: Conservation of the voltage in any loop  $r$

Specifically, we can think of the Feynman diagram as an electrical circuit and associate the momenta with the currents. Thus  $q_j$  are the internal currents flowing along the circuit and  $p_s$  the external currents entering it. When associating the parameters  $\frac{1}{1+v_j}$  with the resistance of the  $j$ th line (so  $v_j$  can be regarded as the conductance of the  $j$ th line), we see explicitly that Eqs. (44) and (45) simply become the Kirchhoff's laws in this circuit analogy. Eq. (44) shows that the sum of “voltage drop” around any closed loop is zero, and Eq. (45) indicates that the sum of “currents” flowing a vertex is zero.

Moreover, if we associate the voltage with the coordinate  $x_\mu$  of the vertex, we can even inquire the physical meaning of Ohm's law:

$$V = IR \quad (46)$$

to be the following relation by translating it into the language of Feynman diagrams:

$$\Delta x_j^\mu = \frac{q_j^\mu}{1 + v_j}, \quad (47)$$

where  $q_j$ ,  $v_j$  are the momentum and UVDP parameter carried on the internal line, and  $\Delta x_j$  is the displacement between two points connected by this line. In fact, Eq. (47) is just

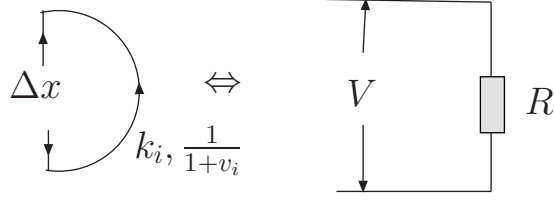


FIG. 9: Ohm's Law

the equation of motion for a free particle, which becomes more apparent in terms of the component forms,

$$\Delta \vec{x}_j = \vec{q}_j \cdot \frac{1}{1+v_j}, \quad \Delta t_j = q_j^0 \cdot \frac{1}{1+v_j}, \quad \frac{\Delta \vec{x}_j}{\Delta t_j} = \frac{\vec{q}_j}{q_j^0}. \quad (48)$$

As the parameter  $v_i$  is positive definite, the causal propagation of the particle is guaranteed:

$$\frac{\Delta t_j}{q_j^0} = \frac{1}{1+v_j} > 0. \quad (49)$$

As the particle goes in the  $\vec{q}_j$  direction according to (48), it moves either forward or backward in time depending on whether the sign of the energy  $q_i^0$  is positive or negative. This agrees with the interpretation of causality of Feynman propagator in QFT.

The above description provides us a physical picture of the circuit analogy which can be summarized as follows

$$\text{Feynman diagrams} \Leftrightarrow \text{Electrical Circuit diagrams} \quad (50)$$

$$\text{Displacement } \Delta x_j \Leftrightarrow \text{Voltage} \quad (51)$$

$$\text{UVDP parameter } v_i \Leftrightarrow \text{Conductance} \geq 0 \quad (52)$$

$$\text{Free particle equation of motion} \Leftrightarrow \text{Ohm's law} \quad (53)$$

$$\text{Cross term cancelation condition for ILIs} \Leftrightarrow \text{Kirchhoff's Law} \quad (54)$$

while the positivity of the UVDP parameter  $v_i$  as the “conductance” is related to the causality of propagation for the free particles.



FIG. 10: Analogue of Feynman diagrams and electrical circuit diagrams

In order to carry out the integral over  $l_r$  in Eq.(42), it is useful to make the quadratic terms of the momentum  $l_r$  diagonal. First write it in terms of the matrix form

$$\sum_{j,r,r'} \frac{\eta_{jr} \eta_{jr'} l_r l_{r'}}{1+v_j} = \sum_{r,r'} l_r M_{rr'} l_{r'} \equiv L^T M L, \quad M_{rr'} = \sum_j \frac{\eta_{jr} \eta_{jr'}}{1+v_j}, \quad (55)$$

where  $L^T = (l_1, \dots, l_k)$  is the transpose of the vector  $L$  and  $M_{rr'}$  is a symmetric matrix. We then diagonalize the matrix  $M$  by an orthogonal transformation  $O$  with

$$L = OL', \quad O^T MO = \text{diag}(\lambda_1, \dots, \lambda_k) \equiv \text{diag}(\lambda_+, \lambda_{-(1)}, \dots, \lambda_{-(k-1)}), \quad (56)$$

where  $\lambda_r$  ( $r = 1, \dots, k$ ) or  $\lambda_+$ ,  $\lambda_{-(r)}$  ( $r = 1, \dots, k-1$ ) are the eigenvalues of the matrix  $M$ , corresponding to the eigenvectors  $L' = (l'_1, \dots, l'_k)^T \equiv (l'_+, l'_{-(1)}, \dots, l'_{-(k-1)})^T$ . As the transformation matrix  $O$  is orthogonal, the integration measure remains unchanged  $d^4 l'_1 \dots d^4 l'_k = d^4 l_1 \dots d^4 l_k$ . Thus, the integral Eq.(42) can be simplified to:

$$\begin{aligned} I(p_1, \dots, p_m) &= \frac{\Gamma(\sum_{j=1}^n \alpha_j)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^n \frac{1}{1+v_j}) \\ &\quad \int d^4 l_1 \dots d^4 l_k \frac{N}{[\sum_{j=1}^n \frac{k_j^2 - m_j^2}{1+v_j}]^{\sum_{j=1}^n \alpha_j}} \\ &= \frac{\Gamma(\sum_{j=1}^n \alpha_j)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^n \frac{1}{1+v_j}) \\ &\quad \int d^4 l'_1 \dots d^4 l'_k \frac{N}{[\sum_{j=1}^n \frac{q_j^2 - m_j^2}{1+v_j} + \sum_r \lambda_r l'^2_r]^{\sum_{j=1}^n \alpha_j}} \end{aligned} \quad (57)$$

For a generic k-loop integrals where  $k \geq 2$  and  $n > k$ , we have the inequality  $\sum_{j=1}^n \alpha_j \geq \frac{k(k+1)}{2} \geq 2k - 1$ .<sup>1</sup> Thus we can explicitly integrate out the loop momenta, as these integrals are already convergent. In particular, when the numerator  $N$  contains no  $l'_i$  terms, we can integrate out the last  $(k-1)$  internal loop momenta, say  $l'_2, l'_3, \dots, l'_k$ .

$$\begin{aligned} I(p_1, \dots, p_m) &= \frac{\Gamma(\sum_{j=1}^n \alpha_j - 2k + 2)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^\infty \prod_{j=1}^n \frac{dv_j}{(1+v_j)^{\alpha_j+1}} \delta(1 - \sum_{j=1}^n \frac{1}{1+v_j}) \prod_{r=1}^{k-1} \frac{1}{\lambda_{-(r)}^2} \\ &\quad \int d^4 l'_+ \frac{1}{[\sum_{j=1}^n \frac{q_j^2 - m_j^2}{1+v_j} + \lambda_+ l'^2_+]^{\sum_{j=1}^n \alpha_j - 2(k-1)}}. \end{aligned} \quad (58)$$

By a rescaling  $l_+ \rightarrow \sqrt{\lambda_+} l'_+$ , we then obtain the following form:

$$\begin{aligned} I(p_1, \dots, p_m) &= \frac{\Gamma(\sum_{j=1}^n \alpha_j - 2k + 2)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^n \frac{1}{1+v_j}) \\ &\quad \frac{1}{(\det |M|)^2} \int d^4 l_+ \frac{1}{[\sum_{j=1}^n \frac{q_j^2 - m_j^2}{1+v_j} + l_+^2]^{\sum_{j=1}^n \alpha_j - 2(k-1)}}, \end{aligned} \quad (59)$$

with the definition of the determinant for the matrix  $M$

$$\det |M| = \prod_{r=1}^k \lambda_r \equiv \lambda_+ \prod_{r=1}^{k-1} \lambda_{-(r)}. \quad (60)$$

---

<sup>1</sup> The first inequality comes from the fact that in order that the expression is generic, we need to consider every type of internal momentum combinations in the denominator, such as  $(l'_j - p_j)^2$ ,  $(l'_i + l'_{i+1} - p_{i(i+1)})^2$ , ...,  $(l'_1 + l'_2 + \dots + l'_k - p_{12\dots k})^2$ . The total number of the combinations is  $k + (k-1) + \dots + 1 = \frac{k(k+1)}{2}$ . If all the types of combinations appear in the denominator, then the inequality holds.

The above expression is the required form of ILIs, where the ILIs for the momentum integral on  $l_+$  reflects the overall divergence of the Feynman diagram. From the above expression, it is clear that the UV divergences contained in the loop momentum integrals over  $l'_{-(r)}$  ( $r = 1, \dots, k-1$ ) for the original loop subdiagrams are now characterized by the possible zero eigenvalues  $\lambda_{-(r)} \rightarrow 0$  ( $r = 1, \dots, k-1$ ) of the matrix  $M$  in the allowed regions of the parameters  $v_i$  ( $i = 1, \dots, n$ ). Namely, each zero eigenvalue  $\lambda_{-(r)} \rightarrow 0$  resulted from some infinities of parameters  $v_i$  in the UVDP parameter space leads to a singularity for the parameter integrals, which corresponds to the divergence of subdiagram in the relevant loop momentum integral.

By applying the general LORE formulae to the above integration over the momentum  $l_+$ , we get:

$$\begin{aligned}
I(p_1, \dots, p_m) &= \frac{\Gamma(\sum_{j=1}^n \alpha_j - 2k + 2)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^n \frac{1}{1+v_j}) \frac{1}{(\det |M|)^2} \\
&\quad \lim_{N, M_l^2} \sum_l^N c_l^N \int d^4 l_+ \frac{i(-1)^{\sum_{j=1}^n \alpha_j}}{[\sum_{j=1}^n \frac{q_j^2 - m_j^2}{1+v_j} + l_+^2 + M_l^2]^{\sum_{j=1}^n \alpha_j - 2(k-1)}} \\
&= \frac{\Gamma(\sum_{j=1}^n \alpha_j - 2k + 2)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^n \frac{1}{1+v_j}) \\
&\quad \frac{1}{(\det |M|)^2} I_{-2\alpha}^R(\mathcal{M}^2)
\end{aligned} \tag{61}$$

with

$$\alpha = \sum_{j=1}^n \alpha_j - 2k, \quad \mathcal{M}^2 = \sum_{j=1}^n (m_j^2 - q_j^2)/(1+v_j) \tag{62}$$

where  $I_{-2\alpha}^R(\mathcal{M}^2)$  is the regularized 1-fold ILI for the possible overall divergence of the Feynman diagram.

In general, there are  $(k-1)$  zero eigenvalues  $\lambda_{-(r)} \rightarrow 0$  ( $r = 1, \dots, k-1$ ) in the UVDP parameter space for the  $k$ -rank matrix  $M$ , and they correspond to the divergences of the  $(k-1)$  loop subdiagrams in the momentum space. In principle, to arrive at the  $k$ -fold ILIs for the  $k$ -loop Feynman diagrams, one may perform  $(n-k-1)$  integrations in the UVDP parameter space. It requires us to appropriately analyze the zero eigenvalues of the matrix  $M$  associated with the corresponding regions of the UVDP parameters. Alternatively, one may make an appropriate parameter transformation, so that the integrations on the  $(n-k-l)$  parameters become convergent for the considered regions of parameters in a new UVDP parameter space, thus they can be integrated safely. As a consequence, we obtain the desired  $k$ -fold ILIs. We shall illustrate in detail its consistency and advantage by applying it to the general  $\alpha\beta\gamma$  integrals in the  $\phi^4$  scalar theory. So far it becomes apparent that the above general procedure explicitly realizes the UVDP parametrization and systematically obtains the ILIs and this shows the powerful advantage when merging the LORE method with the Bjorken-Drell analogy between Feynman diagrams and electrical circuit diagrams.

In order to demonstrate explicitly the correspondence between two kinds of divergences in the UVDP parameter space and in the momentum space, we are going to apply the above general procedure to the  $\alpha\beta\gamma$  integral in next section.

## VI. DIVERGENCE CORRESPONDENCE BETWEEN SUBDIAGRAMS AND UVDP PARAMETERS

The corresponding Feynman diagram for  $\alpha\beta\gamma$  integral is already shown in Fig. (5). With the internal momenta  $k_j$  and the particular choice of loops defined therein, we can rewrite the  $\alpha\beta\gamma$  integral as follows

$$\begin{aligned}
I_{\alpha\beta\gamma} &= \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{(k_1^2 - m_1^2)^\alpha (k_2^2 - m_2^2)^\beta (k_3^2 - m_3^2)^\gamma} \\
&= \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1 + v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^3 \frac{1}{1 + v_j}) \\
&\quad \frac{1}{\left[ \frac{k_1^2 - m_1^2}{1 + v_1} + \frac{k_2^2 - m_2^2}{1 + v_2} + \frac{k_3^2 - m_3^2}{1 + v_3} \right]^{\alpha + \beta + \gamma}}, \tag{63}
\end{aligned}$$

where we have introduced a new notation  $\alpha_i$  ( $i=1,2,3$ ) corresponding to  $\alpha, \beta, \gamma$  in the second line. According to the diagram, we have the momentum conservation, either for overall diagram or for both vertices:

$$p_1 = -p_2 \equiv p,$$

and

$$p_1 - k_1 - k_2 - k_3 = 0, \tag{64a}$$

$$p_2 + k_1 + k_2 + k_3 = 0. \tag{64b}$$

Following Eq. (38), we decompose the internal momenta  $k_j$  into two parts: one represents the loop momentum flowing along the line  $j$ , and the other for the external one carried by  $j$

$$k_1 = q_1 + l_1, \tag{65a}$$

$$k_2 = q_2 + l_2, \tag{65b}$$

$$k_3 = q_3 - l_1 - l_2, \tag{65c}$$

We then arrive at the momentum conservation laws for either vertex in terms of  $q_j$

$$p_1 = q_1 + q_2 + q_3 = -p_2 = p. \tag{66}$$

Replacing the  $k_j$  with  $q_j$  and  $l_r$  in Eq. (63) and changing the integral variables to  $l_r$  give us

$$I_{\alpha\beta\gamma} = \int \frac{d^4 l_1}{(2\pi)^4} \int \frac{d^4 l_2}{(2\pi)^4} \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1 + v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^3 \frac{1}{1 + v_j}) \frac{1}{D^{\alpha + \beta + \gamma}},$$

with

$$D = \sum_{j=1}^3 \frac{q_j^2 - m_j^2}{1 + v_j} + 2\left(\frac{q_1}{1 + v_1} - \frac{q_3}{1 + v_3}\right)l_1 + 2\left(\frac{q_2}{1 + v_2} - \frac{q_3}{1 + v_3}\right)l_2 + L^T M L, \tag{67}$$

where we have introduced the definitions:

$$L \equiv \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \quad M \equiv \begin{pmatrix} \frac{1}{1+v_1} + \frac{1}{1+v_3} & \frac{1}{1+v_3} \\ \frac{1}{1+v_3} & \frac{1}{1+v_2} + \frac{1}{1+v_3} \end{pmatrix}.$$

The elimination of the cross terms in the denominator  $D$  requires that

$$\frac{q_1}{1+v_1} - \frac{q_3}{1+v_3} = 0, \quad (68a)$$

$$\frac{q_2}{1+v_2} - \frac{q_3}{1+v_3} = 0. \quad (68b)$$

These two formula explicitly illustrate the Kirchhoff's law for two loops in the electrical circuit analogy of . By taking into account Eqs. (68) and (66) together, we obtain the solutions:

$$q_1 = \frac{1+v_1}{3+v_1+v_2+v_3} p, \quad (69a)$$

$$q_2 = \frac{1+v_2}{3+v_1+v_2+v_3} p, \quad (69b)$$

$$q_3 = \frac{1+v_3}{3+v_1+v_2+v_3} p. \quad (69c)$$

In order to perform the integral over  $l_r$ , we may first diagonalize the matrix  $M$  by a  $2 \times 2$  orthogonal matrix transformation  $O$ , so that

$$L = OL', \quad O^T MO = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

with  $\lambda_{1,2} = \lambda_{+,-}$  given by

$$\lambda_{\pm} = \frac{(1 + \frac{1}{1+v_3}) \pm \sqrt{(1 + \frac{1}{1+v_3})^2 + 4\Delta}}{2} \quad (70)$$

$$\Delta = \det |M| = \frac{1}{(1+v_1)(1+v_2)} + \frac{1}{(1+v_2)(1+v_3)} + \frac{1}{(1+v_3)(1+v_1)}, \quad (71)$$

which are the two eigenvalues of the matrix  $M$  corresponding to two eigenvectors  $L' = (l'_1, l'_2)$ . Since the transformation matrix  $O$  is orthogonal, the integration measure remains the same  $d^4 l'_1 d^4 l'_2 = d^4 l_1 d^4 l_2$ . Thus, the  $\alpha\beta\gamma$  integral can be reexpressed as:

$$\begin{aligned} I_{\alpha\beta\gamma} &= \int \frac{d^4 l'_1}{(2\pi)^4} \int \frac{d^4 l'_2}{(2\pi)^4} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \\ &\quad \frac{1}{[\sum_{j=1}^3 \frac{q_j^2 - m_j^2}{1+v_j} + \lambda_1 l_1'^2 + \lambda_2 l_2'^2]^{\alpha+\beta+\gamma}} \\ &= \int \frac{d^4 l'_+}{(2\pi)^4} \int \frac{d^4 l'_-}{(2\pi)^4} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \\ &\quad \frac{1}{[\sum_{j=1}^3 \frac{q_j^2 - m_j^2}{1+v_j} + \lambda_+ l_+'^2 + \lambda_- l_-'^2]^{\alpha+\beta+\gamma}}. \end{aligned} \quad (72)$$

From this formalism, it can be shown that the integration over  $l'_-$  represents the integral over subdiagrams, while the one over  $l'_+$  is an overall diagram. However, the matrix  $M$  is not always invertible, since the determinant of  $M$  vanishes when any two of  $v_i$ s tend to  $\infty$ .

More specifically, take  $v_1, v_3 \rightarrow \infty$  for example. In this case the eigenvalue  $\lambda_-$  vanishes. It is also noted that the combination  $\lambda_+ l'_+$  and  $\lambda_- l'_-$  in Eq.(72) are on equal footing in the denominator, it is then expected that  $\lambda_+ l'_+$  and  $\lambda_- l'_-$  approach to infinity at the same speed when both  $l'_\pm \rightarrow \infty$ . Thus, when considering  $\lambda_- \rightarrow 0$  while keeping  $\lambda_+$  finite, it requires that the speed of  $l'_-$  tending to infinity is faster than that of  $l'_+$  in order to keep the balance. Recall that in our previous general discussion on the divergence behavior of overlapping diagrams, one of the features for the subdivergences is that the integration variables approach to infinity faster than the overall one.

Based on the above analysis, we may conclude that the integral over  $l'_-$  reflects the asymptotic behavior of subintegrals when the corresponding UVDP parameters approach to infinity. Here we would like to emphasize that the integration over  $l'_-$  does not correspond to any particular loop in the original Feynman diagram. Rather, it represents all subintegrals and is specified according to the asymptotic regions in the UVDP parameter space. For instance, when the divergences in the UVDP parameter space occur in other regions, such as  $v_1, v_2 \rightarrow \infty$ , then  $l'_-$  reflects the loop composing of lines 1 and 2. The above explicit construction helps us to understand the intuitive analogy between the Feynman diagrams and electrical circuits. Especially, it illustrates why and how the divergences in the subdiagrams are transmitted to the corresponding divergences in the UVDP parameter space. Let us further demonstrate this point from another perspective. By explicitly integrating over  $l'_-$ , we obtain,

$$I_{\alpha\beta\gamma} = \frac{i}{16\pi^2} \int \frac{d^4 l'_+}{(2\pi)^4} \frac{\Gamma(\alpha + \beta + \gamma - 2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \frac{1}{\lambda_-^2} \frac{1}{[\sum_{j=1}^3 \frac{q_j^2 - m_j^2}{1+v_j} + \lambda_+ l'^2_+]^{\alpha+\beta+\gamma-2}} \quad (73)$$

which explicitly shows that when  $\lambda_-$  goes to zero, that is, any two of the three UVDP parameters  $v_i$  approach to infinity, the integrand becomes singular and the integrations over the UVDP parameters give some UV divergences.

By defining a new integral loop momenta  $l_+$  as

$$l_+ \equiv \sqrt{\lambda_+} l'_+, \quad (74)$$

we can transform the  $\alpha\beta\gamma$  integral into a more tractable form:

$$\begin{aligned} I_{\alpha\beta\gamma} &= \frac{i}{16\pi^2} \int \frac{d^4 l_+}{(2\pi)^4} \frac{\Gamma(\alpha + \beta + \gamma - 2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^{\alpha_i+1}} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \\ &\quad \frac{1}{(\det |M|)^2} \frac{1}{[\sum_{j=1}^3 \frac{q_j^2 - m_j^2}{1+v_j} + l_+^2]^{\alpha+\beta+\gamma-2}} \\ &= \frac{i}{16\pi^2} \frac{\Gamma(\alpha + \beta + \gamma - 2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) F(v_k) \\ &\quad \int \frac{d^4 l_+}{(2\pi)^4} \frac{1}{[l_+^2 - \mathcal{M}^2(p^2, m_k^2, v_k)]^{\alpha+\beta+\gamma-2}}, \end{aligned} \quad (75)$$



where

$$\begin{aligned}
F(v_k) &= \frac{(1+v_1)^{1-\alpha}(1+v_2)^{1-\beta}(1+v_3)^{1-\gamma}}{(\det |M|)^2} \\
&= \frac{(1+v_1)^{3-\alpha}(1+v_2)^{3-\beta}(1+v_3)^{3-\gamma}}{(3+v_1+v_2+v_3)^2}, \\
\mathcal{M}^2 &= \sum_{j=1}^3 \frac{m_j^2 - q_j^2}{1+v_j} = \sum_{j=1}^3 \frac{m_j^2}{1+v_j} - \frac{1}{3+v_1+v_2+v_3} p^2.
\end{aligned} \tag{76}$$

The above equation is equivalent to the form in Eq.(25) with a rescaling given in Eq.(28). Nevertheless, the derivation here is more general and systematic and it explicitly shows the advantage when merging the UVDP parametrization and the evaluation of ILIs with the Bjorken-Drell's electrical circuit analogy of the Feynman diagrams.

In general, the integration for the momentum  $l_+$  is divergent and needs to be regularized. By applying the LORE method to the momentum integral, we obtain:

$$\begin{aligned}
I_{\alpha\beta\gamma} &= \frac{i}{16\pi^2} \frac{\Gamma(\alpha+\beta+\gamma-2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) F(v_k) \\
&\quad \lim_{N, M_l^2} \sum_{l=1}^N c_l^N \int \frac{d^4 l_+}{(2\pi)^4} \frac{i(-1)^{\alpha+\beta+\gamma}}{[l_+^2 + M_l^2 + \mathcal{M}^2(p^2, m_k^2, v_k)]^{\alpha+\beta+\gamma-2}} \\
&= \frac{i}{16\pi^2} \frac{\Gamma(\alpha+\beta+\gamma-2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) F(v_k) I_{-2(\alpha+\beta+\gamma-4)}^R(\mathcal{M}^2).
\end{aligned} \tag{77}$$

When applying the above general formula to the case  $\alpha = \gamma = 1, \beta = 2$ , with the similar calculation as the one in the previous section, the result is the same due to the equivalence of Eq. (25) and Eq. (75),

$$\begin{aligned}
I_{121} &= \frac{i}{16\pi^2} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \frac{(1+v_1)^2(1+v_2)(1+v_3)^2}{(3+v_1+v_2+v_3)^2} I_0^R(\mathcal{M}^2) \\
&\rightarrow -\frac{1}{(16\pi^2)^2} \int_0^\infty \prod_{i=1}^3 dv_i \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \frac{1}{(3+v_1+v_2+v_3)^2(1+v_2)} \\
&\quad (\ln \frac{M_c^2}{\mathcal{M}_s^2} - \gamma_\omega + y_0(\frac{\mathcal{M}_s^2}{M_c^2}))
\end{aligned} \tag{78}$$

which shows that the singular behavior in the region  $v_1, v_3 \rightarrow \infty$  becomes obvious as  $\det |M| = \Delta = 0$  due to the zero eigenvalue  $\lambda_- \rightarrow 0$ . For the other two regions:  $v_1, v_2 \rightarrow \infty$  and  $v_2, v_3 \rightarrow \infty$ , the additional factor  $\frac{1}{(1+v_2)}$  in these two cases makes the integration finite. In contrast, for the case  $\alpha = \beta = \gamma = 1$ , there is no such a factor, so that there are more UV divergent structures in all the three regions and is going to be discussed in detail below.

## VII. TREATMENT OF OVERLAPPING DIVERGENCE AND ADVANTAGE OF THE LORE METHOD MERGING WITH BJORKEN-DRELL'S ANALOGY

This section shows that the LORE method merging with the Bjorken-Drell's circuit analogy has the advantage in analyzing the more complicated and challenging overlapping divergence structures of Feynman diagrams. For an explicit demonstration, we are going to

consider the case with  $\alpha = \beta = \gamma = 1$  in the  $\alpha\beta\gamma$  integral. The difficulty lies not only in the quadratic divergence but also in the more complicated overlapping divergence structure. It will be seen that the LORE method merging with Bjorken-Drell's analogy is extremely powerful in unraveling the overlapping divergences.

The general form of  $\alpha\beta\gamma$  integral (Eq.75) can be simplified to:

$$\begin{aligned}
I_{111} &= \frac{i}{16\pi^2} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \frac{1}{(\det |M|)^2} \\
&\quad \lim_{N, M_l^2} \sum_{l=1}^N c_l \int \frac{d^4 l_+}{(2\pi)^4} \frac{-i}{\sum_{j=1}^3 \frac{q_j^2 - m_j^2}{1+v_j} + l_+^2 + M_l^2} \\
&\rightarrow \frac{1}{(16\pi^2)^2} \int_0^\infty \prod_{i=1}^3 \frac{dv_i}{(1+v_i)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \frac{\prod_{j=1}^3 (1+v_j)^2}{(3+v_1+v_2+v_3)^2} \\
&\quad [M_c^2 - \mathcal{M}^2 (\ln \frac{M_c^2}{\mathcal{M}^2} - \gamma_\omega + 1)], \tag{79}
\end{aligned}$$

where we have regularized the overall quadratic divergence of loop momentum integral by the LORE method. The mass factor  $\mathcal{M}$  is given in Eq.(??). The UVDP parameter integrals are more involved due to the appearance of the overlapping divergences. From the expression of integral  $I_{111}$ , it is seen that the three subintegrals  $\alpha\gamma$ ,  $\beta\gamma$ , and  $\alpha\beta$  are all divergent. With the analogy of circuits, we have shown that the UV divergences arising from the large internal loop momenta transmit to the asymptotic regions of UVDP parameter space, where the divergent conductances correspond to the following asymptotic regions in the circuits:

*subdivergence in  $\alpha\beta\gamma$  diagrams  $\Leftrightarrow$  divergence in UVDP parameter space*

$$\text{Circuit 1 : } \alpha\gamma \text{ divergence} \Leftrightarrow v_1 \rightarrow \infty, v_3 \rightarrow \infty, v_2 \rightarrow 0, \tag{80}$$

$$\text{Circuit 2 : } \beta\gamma \text{ divergence} \Leftrightarrow v_2 \rightarrow \infty, v_3 \rightarrow \infty, v_1 \rightarrow 0, \tag{81}$$

$$\text{Circuit 3 : } \alpha\beta \text{ divergence} \Leftrightarrow v_1 \rightarrow \infty, v_2 \rightarrow \infty, v_3 \rightarrow 0. \tag{82}$$

This result can also be obtained by considering the singularities in the determinant  $\det |M| = \Delta$  as discussed in the previous section.

Note that Eq.(79) has a permutation  $Z_3$  symmetry among the three pairs of parameters  $(v_1, m_1)$ ,  $(v_2, m_2)$ ,  $(v_3, m_3)$ , so the treatment on three asymptotic regions in the circuits is essentially the same. Let us consider in detail the first case of the Circuit 1.

**Circuit 1:**  $v_1 \rightarrow \infty$ ,  $v_3 \rightarrow \infty$  and  $v_2 \rightarrow 0$ . In such region, the integral domain can be written as  $\int_V^\infty dv_1 \int_V^\infty dv_3$  with  $\mathcal{M}^2 \rightarrow m_2^2$  and  $F(v_j) \rightarrow \frac{(1+v_1)^2(1+v_3)^2}{(v_1+v_3)^2}$ . Thus the integration is simplified to:

$$\begin{aligned}
I_{111}^{(0)(\alpha\gamma)} &\simeq \frac{1}{(16\pi^2)^2} \int_V^\infty \frac{dv_1}{(1+v_1)^2} \int_V^\infty \frac{dv_3}{(1+v_3)^2} \frac{(1+v_1)^2(1+v_3)^2}{(v_1+v_3)^2} \\
&\quad [M_c^2 - m_2^2 (\ln \frac{M_c^2}{m_2^2} - \gamma_\omega + 1)] \\
&= \frac{1}{(16\pi^2)^2} [M_c^2 - m_2^2 (\ln \frac{M_c^2}{m_2^2} - \gamma_\omega + 1)] \int_V^\infty dv_1 \frac{1}{v_1 + V} + \dots \\
&= \frac{1}{(16\pi^2)^2} [M_c^2 - m_2^2 (\ln \frac{M_c^2}{m_2^2} - \gamma_\omega + 1)] (\ln \frac{M_c^2}{2q_o^2 V} - \gamma_\omega) + \dots \tag{83}
\end{aligned}$$

Note that in the last step, we have applied the LORE method with the treatment discussed in Eq.(32). The dots represent other terms, such as single logarithmic divergent term and finite terms, which are irrelevant to our discussions as our main purpose here is to check the cancelation of the harmful divergences. Note that our result here is factorizable.

In order to compare the above divergence structure with those contained in the subdiagram  $(\alpha\gamma)$ , we calculate the counterterm diagram  $I_{111}^{(c)(\alpha\gamma)}$ :

$$\begin{aligned} I_{111}^{(c)(\alpha\gamma)} &= - \int \frac{d^4 k_2}{(2\pi)^4} \frac{1}{k_2^2 - m_2^2} \text{DP} \left\{ \int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{(k_1^2 - m_1^2)[k_3^2 - m_3^2]} \right\} \\ &\rightarrow - \frac{1}{(16\pi^2)^2} (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) [M_c^2 - m_2^2 (\ln \frac{M_c^2}{m_2^2} - \gamma_\omega + 1)], \end{aligned} \quad (84)$$

where  $\text{DP}\{\dots\}$  means the divergence part of the integral in the bracket, and  $\mu^2$  is the renormalization scale. It is then manifest that when we choose  $\mu^2 = 2q_o^2 V$ , the harmful divergence parts in the two expressions cancel exactly.

With a similar discussion based on the permutation  $Z_3$  symmetry, it is easy to show that the harmful divergent parts in the Circuit 2 and Circuit 3 also cancel exactly. So far we prove that there is no harmful divergence for the case  $\alpha = \beta = \gamma = 1$  when combining with the corresponding counterterm diagrams. We would like to mention that this is different from the dimensional regularization, that in that we have an extra term corresponding to the quadratic divergence  $M_c^2$ . As emphasized in [3], this term is natural to maintain the correct divergent behavior of the original diagram, which can play an important role in effective field theory for obtaining the correct gap equation to describe the dynamically generated spontaneous symmetry breaking[10]. It will explicitly be shown below that the presence of this term prevents us from having a mass independent renormalization scheme. Thus, a consistent renormalization with a well-defined subtraction scheme must be proposed for the LORE method. We propose the following subtraction scheme:

- (i) For quadratic divergence  $(M_c^2 - \mathcal{M}^2)$ , subtract  $(M_c^2 - \mu^2)$  and leave  $(\mu^2 - \mathcal{M}^2)$  in the finite expression;
- (ii) For logarithmic divergence  $(\log \frac{M_c^2}{\mathcal{M}^2} - \gamma_\omega)$ , subtract  $(\log \frac{M_c^2}{\mu^2} - \gamma_\omega)$  and leave term  $\log \frac{\mu^2}{\mathcal{M}^2}$  in the finite expression.

Such a scheme may be regarded as a kind of energy scale subtraction scheme at  $\mu^2$  and is similar to the usual momentum subtraction. For the logarithmic divergence, it appears to be a  $\bar{M}S$ -like scheme in the dimensional regularization as it is associated with the Euler number  $\gamma_w = \gamma_E$ . It is interesting to note that once the energy scale subtraction scheme for both the quadratic and logarithmic terms is set up at the one-loop level with a correlated form  $(M_c^2 - \mu^2)$  and  $\ln M_c^2/\mu^2$  via a single subtracted energy scale  $\mu^2$ , and suppose that the correlated form with a single subtracted energy scale  $\mu^2$  is required to be maintained, thus either the rescaling  $\mu^2 \rightarrow e^{\alpha_0} \mu^2$  or shifting  $\mu^2 \rightarrow \mu^2 - \alpha_0 m^2$  for the subtracted energy scale  $\mu^2$  will not be allowed. As a consequence, the mass renormalization at higher loop becomes well-defined through such an energy scale subtraction scheme at the one loop level, namely fixing the correlated form for the quadratic and logarithmic terms via a single subtracted energy scale  $\mu^2$ .

Based on the above analysis and discussions, we arrive at the following theorems:

*Factorization Theorem for Overlapping Divergences:* Overlapping divergences which contain divergences of subintegrals and overall one in the general Feynman loop integrals become factorizable in the corresponding asymptotic regions.

*Subtraction Theorem for Overlapping Divergences:* For general scalar-type two-loop integral  $I_{\alpha\beta\gamma}$ , when combined with the corresponding subtraction integrals (which is composed

of divergent subintegrals multiplied by an overall integral), the sum will only contain harmless divergence.

For completeness, we have the following theorems for dealing with the Feynman integrals which do not involve the overlapping divergence. They are so obvious that the proofs are omitted here.

*Harmless Divergence Theorem:* If the general loop integral contains no divergent subintegrals, then it is only possible to contain a harmless single divergence arising from the overall divergence.

*Trivial Convergence Theorem:* If the general loop integral contains neither the overall divergence nor the divergent subintegrals, then it is convergent.

In summary, the LORE method can properly deal with the overlapping divergences, especially the subdivergences which is transformed appropriately into the divergences in the UVDP parameter space. To extract them, we need to explore the integrals in different asymptotic regions of the parameter space. Moreover, we demonstrate that these overlapping divergences can well be treated by the LORE method when merging with the Bjorken-Drell's analogy between general Feynman diagrams and electrical circuits, especially the correspondence between the UVDP parameters and the conductances of internal lines in the circuit analogy. By applying this intuitive picture, we can immediately recognize how a divergence in the region of UVDP parameter space corresponds to a certain original divergent subintegral composed by the lines that the divergent UVDP parameters are attached on. This correspondence also helps us to find the right counterterm diagram to cancel the notorious harmful divergences. As a result, we are left with only the finite terms and the harmless divergence which can be absorbed into the overall counterterm at two-loop order. These results are summarized in the four theorems presented above. It is interesting to note that the extension of the LORE method to the calculations beyond two-loop order is straightforward, although we are aware that the degree of complication and difficulty increases dramatically with the increase of loop orders as more and more Feynman diagrams are involved. Nevertheless, it is clearly indicated that merging with the Bjorken-Drell's analogy between Feynman diagrams and electrical circuit diagrams, the LORE method gets its consistency and advantage in the multiloop calculations, especially with the aid of computer.

## VIII. APPLICATION TO $\phi^4$ THEORY AT TWO-LOOP ORDER

The discussion and analysis in the previous sections on the general two-loop integrals appear to be a little bit too abstract. In this section, we shall take the simple scalar  $\phi^4$  theory as a concrete example to illustrate the LORE method in a practical calculation, and leave another application involving tensor-type integrals to a separate paper[22].

The Lagrangian density for  $\phi^4$  theory is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \quad (85)$$

Its Feynman rules may be found in the standard textbooks, such as [16, 17]. Our main purpose here is to explicitly calculate the two-loop contributions to the mass term and coupling constant from the self-energy and vertex diagrams. From this practical calculation, we will demonstrate in detail the consistency and advantage of the LORE method when merging with the Bjorken-Drell's circuits analogy.

### A. Renormalization At One-Loop Level

Before proceeding to a detailed calculation at two-loop level, we need the one-loop counterterms first. This is equivalent to specify the renormalization condition in the LORE method, which is the main goal of this subsection.

At the one loop level, there are two types of diagrams corresponding to the self-energy correction and vertex correction respectively, as shown in Figs. (11) and (12).

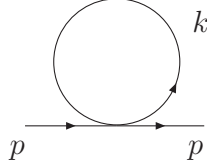


FIG. 11:

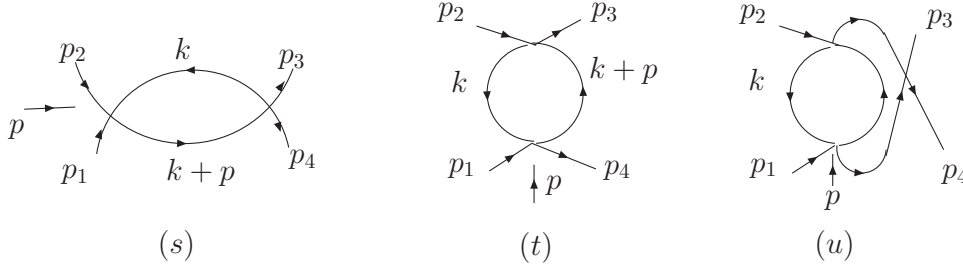


FIG. 12:

For the self-energy correction, the calculation is straightforward and the result is given by

$$\begin{aligned}
 -iM_{(1)}^2 &= -i\lambda \cdot \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\
 &\rightarrow -\frac{i\lambda}{2(4\pi)^2} [M_c^2 - m^2 (\ln \frac{M_c^2}{m_s^2} - \gamma_\omega + 1 + y_2(\frac{m_s^2}{M_c^2}))] \\
 &\rightarrow -\frac{i\lambda}{2(4\pi)^2} [(M_c^2 - m^2) - m^2 (\ln \frac{M_c^2}{m^2} - \gamma_\omega)]
 \end{aligned} \tag{86}$$

where we have applied the LORE method to obtain the result in the second line. The result in the last line is obtained by taking  $\mu_s = 0$  and  $M_c \rightarrow \infty$ . Under the energy scale  $\mu^2$  subtraction scheme described in the previous section, the mass and wave function counterterms take the following forms:

$$-i\delta_{m^2}^{(1)} = \frac{i\lambda}{2(4\pi)^2} [(M_c^2 - \mu^2) - m^2 (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)], \tag{87}$$

$$i\delta_Z^{(1)} = 0, \tag{88}$$

and the finite term is found to be

$$-iM_{(1)}^2 = -\frac{i\lambda}{2(4\pi)^2}[(\mu^2 - m^2) - m^2 \ln \frac{\mu^2}{m^2}]. \quad (89)$$

Note that the result given in Eq.(89) is different from the one obtained by using the dimensional regularization method. The difference arises from the quadratic behavior  $\mu^2$  in the renormalization counterterm. This difference may have important physical implications: it greatly changes the renormalization group[12–14] with a power law running, and generates the physically meaningful dynamical mass scales in the effective field theory[3, 10].

By a similar calculation of the one-loop four-point Green function, we obtain the following vertex correction for s-channel:

$$\begin{aligned} -i\Lambda_{(1)}^{(s)} &= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \\ &= \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{[xk^2 + (1-x)(k+p)^2 - m^2]^2} \\ &= \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{[k^2 + x(1-x)p^2 - m^2]^2} \\ &\rightarrow \frac{i\lambda^2}{2(4\pi)^2} \int_0^1 dx [\ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega], \end{aligned} \quad (90)$$

where  $-p^2 = -(p_1 + p_2)^2 \equiv s$ . For other two channels (t- and u-channels), we will obtain the same expression except for the definition of  $p^2$ :  $-p^2 = -(p_1 - p_4)^2 \equiv t$  for t-channel and  $-p^2 = -(p_1 - p_3)^2 \equiv u$  for u-channel.

According to the renormalization scheme of the LORE method, we have the following counterterm:

$$-i\delta_\lambda^{(1)} = -\frac{3i\lambda^2}{2(4\pi)^2} [\ln \frac{M_c^2}{\mu^2} - \gamma_\omega], \quad (91)$$

where the factor 3 comes from three diagrams corresponding to the  $s$ ,  $t$ ,  $u$ -channels.

So the renormalized vertex correction is simply given by:

$$-i\Lambda_{(1)} = \frac{i\lambda^2}{2(4\pi)^2} \int_0^1 dx [\ln \frac{\mu^2}{m^2 + x(1-x)s} + \ln \frac{\mu^2}{m^2 + x(1-x)t} + \ln \frac{\mu^2}{m^2 + x(1-x)u}]. \quad (92)$$

## B. Self-Energy Contribution at Two Loop

There are two diagrams contributing to the two-loop self-energy corrections, which are shown in Figs. (13) and (14).

The calculation of diagram (a) is straightforward and the result is:

$$\begin{aligned} -iM_{(2)}^{2(a)} &= \frac{1}{4}(-i\lambda)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{i}{k_1^2 - m^2} \frac{i}{k_1^2 - m^2} \int \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_2^2 - m^2} \\ &\rightarrow \frac{1}{4} \frac{i\lambda^2}{(16\pi)^2} (\ln \frac{M_c^2}{m^2} - \gamma_\omega) \cdot [M_c^2 - m^2 (\frac{M_c^2}{m^2} - \gamma_\omega + 1)]. \end{aligned} \quad (93)$$

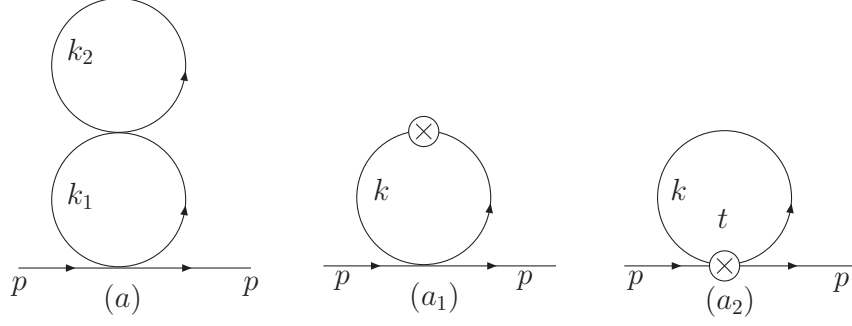


FIG. 13:

The corresponding counterterm diagrams are shown in  $(a_1)$  and  $(a_2)$ , and their sum gives:

$$\begin{aligned}
-iM_{(2)}^{2(a_1)+(a_2)} &= \frac{1}{2}(-i\lambda)(-i\delta_{m^2}) \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{(k^2 - m^2)^2} + \frac{1}{2}(-i\delta_\lambda^t) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\
&\rightarrow -\frac{1}{4} \frac{i\lambda^2}{(16\pi^2)^2} \{[(M_c^2 - \mu^2) - m^2(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)](\ln \frac{M_c^2}{m^2} - \gamma_\omega) \\
&\quad + (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) \cdot [M_c^2 - m^2(\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1)]\}, \tag{94}
\end{aligned}$$

where  $\delta_{m^2}$  is the one-loop mass counterterm defined in Eq. (87) and  $\delta_\lambda^t$  only the t-channel vertex counterterm, which is represented in Fig.13( $a_2$ ). Thus, the sum of  $(a)$ ,  $(a_1)$  and  $(a_2)$  gives:

$$\begin{aligned}
-iM_{(2)}^{2(a)+(a_1)+(a_2)} &= \frac{1}{4} \frac{i\lambda^2}{(16\pi^2)^2} \{-(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)[(M_c^2 - \mu^2) - m^2(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)] \\
&\quad + [(\mu^2 - m^2) - m^2 \ln \frac{\mu^2}{m^2}] \ln \frac{\mu^2}{m^2}\}. \tag{95}
\end{aligned}$$

According to our renormalization scheme proposed in the previous subsection, the overall two-loop counterterm for diagram  $(a)$  is defined as:

$$-i\delta_{m^2}^{(a)} = \frac{1}{4} \frac{i\lambda^2}{(16\pi^2)^2} (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) [(M_c^2 - \mu^2) - m^2(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)], \tag{96}$$

and the renormalized correction to the two-point function from this diagram is:

$$-iM_{(2)R}^{2(a)} = \frac{1}{4} \frac{i\lambda^2}{(16\pi^2)^2} [(\mu^2 - m^2) - m^2 \ln \frac{\mu^2}{m^2}] \ln \frac{\mu^2}{m^2}. \tag{97}$$

Let us now compute the most complicated diagrams in  $\phi^4$  theory at two loop order, namely the sunrise diagram. According to the internal momentum parameterizations shown in the diagram  $(b)$  and the general Feynman rules, we can write the expression explicitly:

$$\begin{aligned}
-iM_{(2)}^{2(b)} &= \frac{1}{6}(-i\lambda)^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{i}{(k_1 + p)^2 - m^2} \frac{i}{k_2^2 - m^2} \frac{i}{(k_1 + k_2)^2 - m^2} \\
&= \frac{i\lambda^2}{6} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{1}{[(k_1 + p)^2 - m^2](k_2^2 - m^2)[(k_1 + k_2)^2 - m^2]}. \tag{98}
\end{aligned}$$

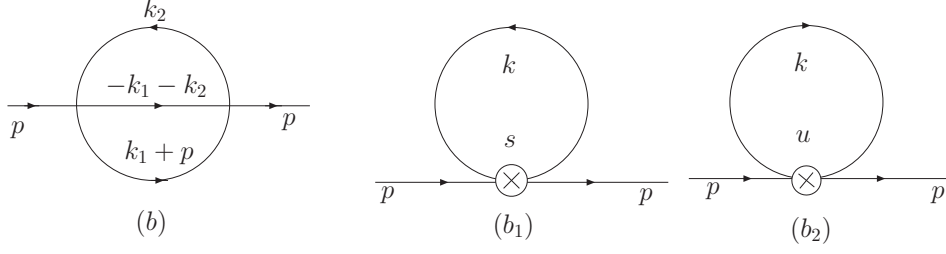


FIG. 14:

Obviously, the above integral is just the special case of  $\alpha\beta\gamma$  integral with  $\alpha = \beta = \gamma = 1$  and the same mass  $m_i^2 = m^2$ . Thus we can apply the general result Eq. (79) to this case

$$\begin{aligned}
 -iM_{(2)}^{2(b)} &\rightarrow \frac{i\lambda^2}{6(16\pi^2)^2} \int_0^\infty \prod_{j=1}^3 \frac{dv_j}{(1+v_j)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \frac{\prod_{j=1}^3 (1+v_j)^2}{(3+v_1+v_2+v_3)^2} \\
 &\quad [M_c^2 - \mathcal{M}^2 (\ln \frac{M_c^2}{\mathcal{M}^2} - \gamma_\omega + 1)],
 \end{aligned} \tag{99}$$

with

$$\mathcal{M}^2 = m^2 - \frac{1}{3+v_1+v_2+v_3} p^2.$$

We now compute the contribution to the two-point Green function, rather than just giving the asymptotic expression for showing the cancelation of harmful divergences, which was already demonstrated in the previous section. For this purpose, it is useful to introduce a new set of UVDP parameters  $u, v, w$  via

$$\frac{1}{1+v_1} \equiv \frac{1}{(1+u)(1+w)}, \tag{100a}$$

$$\frac{1}{1+v_2} \equiv \frac{u}{1+u}, \tag{100b}$$

$$\frac{1}{1+v_3} \equiv \frac{1}{(1+u)(1+v)}, \tag{100c}$$

so the integration measure is now

$$\begin{aligned}
 &\int_0^\infty \prod_{j=1}^3 \frac{dv_j}{(1+v_j)^2} \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \\
 &= \int_0^\infty \frac{du}{(1+u)^3} \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta(1 - \frac{1}{1+w} - \frac{1}{1+v}),
 \end{aligned} \tag{101}$$

with

$$\mathcal{M}^2 = m^2 - \frac{u}{1+u} \frac{1}{u(1+w)(1+v)+1} p^2, \tag{102}$$

$$F = \frac{(1+u)^4}{[u + \frac{1}{(1+w)(1+v)}]^2}. \tag{103}$$



With the above transformation, we finally arrive at:

$$-iM_{(2)}^{2(b)} = \frac{i\lambda^2}{6(16\pi^2)^2} \int_0^\infty du \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta\left(1 - \frac{1}{1+w} - \frac{1}{1+v}\right) \frac{1+u}{\left[u + \frac{1}{(1+w)(1+v)}\right]^2} [M_c^2 - \mathcal{M}^2(\ln \frac{M_c^2}{\mathcal{M}^2} - \gamma_\omega + 1)]. \quad (104)$$

For the quadratic divergence, we can carry out the integration:

$$\begin{aligned} -iM_{(2)quad}^{2(b)} &= \frac{i\lambda^2}{6(16\pi^2)^2} M_c^2 \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta\left(1 - \frac{1}{1+w} - \frac{1}{1+v}\right) \\ &\quad \int_0^\infty du \frac{1+u}{\left[u + \frac{1}{(1+w)(1+v)}\right]^2} \\ &= \frac{i\lambda^2}{6(16\pi^2)^2} M_c^2 \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta\left(1 - \frac{1}{1+w} - \frac{1}{1+v}\right) \\ &\quad \int_0^\infty du \left[ \frac{1}{u + \frac{1}{(1+w)(1+v)}} + \frac{1}{\left[u + \frac{1}{(1+w)(1+v)}\right]^2} \left(1 - \frac{1}{(1+w)(1+v)}\right) \right] \\ &\rightarrow \frac{i\lambda^2}{6(16\pi^2)^2} M_c^2 [3(\ln \frac{M_c^2}{q_o^2} - \gamma_\omega) + 1] \end{aligned} \quad (105)$$

which is local when we choose the free scale  $q_o^2 = \mu^2$ .

For the logarithmic divergence part, the result is given by:

$$\begin{aligned} -iM_{(2)log}^{2(b)} &= -\frac{i\lambda^2}{6(16\pi^2)^2} \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta\left(1 - \frac{1}{1+w} - \frac{1}{1+v}\right) \int_0^\infty du \frac{1+u}{\left[u + \frac{1}{(1+w)(1+v)}\right]^2} \\ &\quad \left\{ m^2 - \frac{u}{(1+u)(1+w)(1+v)\left[u + \frac{1}{(1+w)(1+v)}\right]} p^2 \right\} \left[ \ln \frac{M_c^2}{\mathcal{M}^2} - \gamma_\omega + 1 \right] \end{aligned} \quad (106)$$

From the analysis and discussion presented in previous sections for the overlapping divergences, there are three parameter regions which contain divergent contributions. To extract them, we need to separate the general expression into several parts, each of which may give an asymptotical result in a single region. In terms of the new set of UVDP parameters  $u, v, w$ , the situation becomes much simpler than the original parameters  $v_1, v_2, v_3$ . The coefficients of the logarithmic divergence in the above expression can be separated into the following four parts:

$$\begin{aligned} &\frac{1+u}{\left[u + \frac{1}{(1+w)(1+v)}\right]^2} \left\{ m^2 - \frac{u}{(1+u)(1+w)(1+v)\left[u + \frac{1}{(1+w)(1+v)}\right]} p^2 \right\} \\ &= \frac{1}{u + \frac{1}{(1+w)(1+v)}} m^2 + \frac{1}{\left[u + \frac{1}{(1+w)(1+v)}\right]^2} \left(1 - \frac{1}{(1+w)(1+v)}\right) m^2 \\ &\quad - \frac{u}{(1+w)(1+v)\left[u + \frac{1}{(1+w)(1+v)}\right]^3} p^2 \\ &= I + (II + III) + IV. \end{aligned} \quad (107)$$

These four parts are divergent in the following asymptotic UVDP parameter regions:

$$\begin{aligned} I : & \quad u \rightarrow \infty \quad vw = 1; \quad v_1 \rightarrow \infty \quad v_3 \rightarrow \infty \quad v_2 \rightarrow 0 \\ II : & \quad v \rightarrow \infty \quad u \rightarrow 0 \quad w \rightarrow 0; \quad v_2 \rightarrow \infty \quad v_3 \rightarrow \infty \quad v_1 \rightarrow 0 \\ III : & \quad w \rightarrow \infty \quad u \rightarrow 0 \quad v \rightarrow 0; \quad v_2 \rightarrow \infty \quad v_1 \rightarrow \infty \quad v_3 \rightarrow 0 \\ IV : & \quad p^2 \gg m^2, \end{aligned}$$

where the second part in Eq.(107) contains two asymptotic regions II and III symmetric under the exchange of parameters  $v$  and  $w$  or  $v_1$  and  $v_3$ . In general, it is difficult to carry out the whole integration and obtaining a complete result for  $-iM_{(2)}^{2(b)}$  due to the complicity of  $\mathcal{M}^2$  in the logarithm. But it is sufficient for our present purpose to obtain  $-iM_{(2)}^{2(b)}$  by simplifying  $\mathcal{M}^2$  in the above asymptotic regions, allowing us to get the results up to logarithmic divergence.

**Region (I):**  $u \rightarrow \infty$ . In this region,  $\mathcal{M}^2$  can be simplified to the form:

$$\mathcal{M}^2 \simeq m^2. \quad (108)$$

So the approximate expression is given by:

$$\begin{aligned} -iM_{(2)log}^{2(b)(I)} &\simeq -\frac{i\lambda^2}{6(16\pi^2)^2} \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta(1 - \frac{1}{1+w} - \frac{1}{1+v}) \\ &\quad \int_0^\infty \frac{du}{u + \frac{1}{(1+w)(1+v)}} m^2 (\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1) \\ &\rightarrow -\frac{i\lambda^2 m^2}{6(16\pi^2)^2} (\ln \frac{M_c^2}{q_o^2} - \gamma_\omega + 2) (\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1). \end{aligned} \quad (109)$$

Notice that in the case  $u \rightarrow \infty$ , the only relevant mass scale is  $m^2$ , so we can simply take  $q_o^2 = m^2$ . The final expression in this region is found to be:

$$-iM_{(2)log}^{2(b)(I)} \simeq -\frac{i\lambda^2}{6(16\pi^2)^2} m^2 [(\ln \frac{M_c^2}{m^2} - \gamma_\omega)^2 + 3(\ln \frac{M_c^2}{m^2} - \gamma_\omega)]. \quad (110)$$

**Region (II+III):**  $v \rightarrow \infty$  or  $w \rightarrow \infty$ . It is interesting to note that the integral is symmetric under the exchange of parameters  $v$  and  $w$ . Thus taking the limit  $v \rightarrow \infty$ , or  $w \rightarrow \infty$ , we shall arrive at the same results. In both cases, the asymptotic form of  $\mathcal{M}^2$  is given by:

$$\mathcal{M}^2 \simeq m^2 \quad (111)$$

$$\begin{aligned} -iM_{(2)log}^{2(b)(II+III)} &\simeq -\frac{i\lambda^2}{6(16\pi^2)^2} m^2 \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta(1 - \frac{1}{1+w} - \frac{1}{1+v}) \\ &\quad [1 - \frac{1}{(1+w)(1+v)}] \int_0^\infty \frac{du}{[u + \frac{1}{(1+w)(1+v)}]^2} [\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1] \\ &= -\frac{i\lambda^2}{6(16\pi^2)^2} m^2 \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta(1 - \frac{1}{1+w} - \frac{1}{1+v}) \\ &\quad [1 - \frac{1}{(1+w)(1+v)}] (1+w)(1+v) (\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1) \\ &= -\frac{i\lambda^2}{6(16\pi^2)^2} m^2 [\int_0^\infty \frac{dw}{1+w} + \int_0^\infty \frac{dv}{1+v} - 1] (\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1) \\ &\rightarrow -\frac{i\lambda^2}{6(16\pi^2)^2} m^2 [2(\ln \frac{M_c^2}{q_o^2} - \gamma_\omega) - 1] (\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1), \end{aligned} \quad (112)$$

where in the third equality, we have used the constraint in the delta function  $(1+w)(1+v) = (1+v) + (1+w)$  to simplify the integral into the form of two 1-loop ILIs which can be

regularized by the LORE method as shown previously. Again the only mass scale in the limit  $v \rightarrow \infty$  or  $w \rightarrow \infty$  is the mass of the particle  $m^2$ , so the scale  $q_o^2$  can be fixed to be  $m^2$  and the result is given by:

$$-iM_{2\log}^{2(b)(II+III)} \simeq -\frac{i\lambda^2}{6(16\pi^2)^2} m^2 [2(\ln \frac{M_c^2}{m^2} - \gamma_\omega)^2 + (\ln \frac{M_c^2}{m^2} - \gamma_\omega)]. \quad (113)$$

Note that in  $-iM_{(2)\log}^{2(b)(I)}$  and  $-iM_{(2)\log}^{2(b)(II+III)}$ , there are three logarithmic divergences hidden in the UVDP parameter space, and they reproduce the corresponding subdivergences in the subdiagrams of Fig.13(b). This feature was already anticipated by the electric circuits analogy of Feynman diagrams discussed in section III. However, when adopting a different set of UVDP parameters  $u, v, w$  transformed from the ones  $v_1, v_2, v_3$ , the divergence regions in the parameter space are also changed correspondingly.

**Region (IV):**  $-p^2 \gg m^2$ . In this region, we obtain the first order correction to the wave function renormalization in the  $\phi^4$  theory. Clearly, there is no harmful divergence in this region as all the integrals of UVDP parameters are convergent. When  $-p^2 \gg m^2$ , we can ignore all the terms proportional to  $m^2$  in  $\mathcal{M}^2$  and the integral can be simplified to:

$$\begin{aligned} -iM_{(2)\log}^{2(b)(IV)} &\simeq -\frac{i\lambda^2}{6(16\pi^2)^2} (-p^2) \int_0^\infty \frac{dw}{(1+w)^3} \frac{dv}{(1+v)^3} \delta(1 - \frac{1}{1+w} - \frac{1}{1+v}) \\ &\quad \int_0^\infty \frac{duu}{[u + \frac{1}{(1+w)(1+v)}]^3} [\ln \frac{M_c^2}{(1+u)[u(1+w)(1+v)+1]} (-p^2) - \gamma_\omega + 1] \\ &= \frac{i\lambda^2}{6(16\pi^2)^2} p^2 [\frac{1}{2} (\ln \frac{M_c^2}{-p^2} - \gamma_\omega + 1) \\ &\quad + \frac{1}{108} (-81 - 2\psi^{(1)}(\frac{1}{6}) - 2\psi^{(1)}(\frac{1}{3}) + 2\psi^{(1)}(\frac{2}{3}) + 2\psi^{(1)}(\frac{5}{6}))], \end{aligned} \quad (114)$$

where  $\psi^{(1)}(z) \equiv \frac{d^2}{dz^2} \ln \Gamma(z)$  is the polygamma function of order 1.

Adding up all the contributions Eqs.(105),(110)(113),(114), we arrive at the final result for the divergent contributions of the sunrise diagram Fig.13(b)

$$\begin{aligned} -iM_{(2)}^{2(b)} &\simeq \frac{i\lambda^2}{6(16\pi^2)^2} \{ [3(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) + 1] M_c^2 - 3m^2 (\ln \frac{M_c^2}{m^2} - \gamma_\omega)^2 - 4m^2 (\ln \frac{M_c^2}{m^2} - \gamma_\omega) \\ &\quad + \frac{1}{2} p^2 (\ln \frac{M_c^2}{-p^2} - \gamma_\omega) \}. \end{aligned} \quad (115)$$

The counterterm diagram for Fig.13(b) is shown in Fig. 13(b<sub>1,2</sub>) and the result is simply given by:

$$\begin{aligned} -iM_{(2)}^{2(b_1)+(b_2)} &= \frac{1}{2} (-i\delta_\lambda^{s+u}) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\ &\rightarrow -\frac{i\lambda^2}{2(16\pi^2)^2} (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) [M_c^2 - m^2 (\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1)]. \end{aligned} \quad (116)$$

Note that we have used the vertex counterterm insertion of  $s$  and  $u$ -channels, so there is a factor of 2 in the above calculation. By summing up Figs. 13 (b) and (b<sub>1</sub>), we obtain:

$$\begin{aligned} -iM_{(2)}^{2(b)+(b_1)+(b_2)} &= \frac{i\lambda^2}{(16\pi^2)^2} \{ [\frac{1}{6} (M_c^2 - \mu^2) - \frac{1}{6} m^2 (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) + \frac{1}{12} p^2 (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)] \\ &\quad + [\frac{1}{6} (\mu^2 - m^2) - \frac{1}{2} m^2 (\ln \frac{\mu^2}{m^2})^2 - \frac{2}{3} m^2 \ln \frac{\mu^2}{m^2} + \frac{1}{12} p^2 \ln \frac{\mu^2}{-p^2}] \} + \dots \end{aligned} \quad (117)$$

As it is expected, the potentially harmful divergences  $m^2(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) \ln \frac{\mu^2}{m^2}$  cancel exactly.

By considering the following overall counterterms for diagram (b):

$$i(p^2 \delta_Z^{(b)} - \delta_{m^2}^{(b)}) = -\frac{i\lambda^2}{(16\pi^2)^2} [\frac{1}{6}(M_c^2 - \mu^2) - \frac{1}{6}m^2(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) + \frac{1}{12}p^2(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)], \quad (118)$$

we get the final contribution to the two-loop self-energy as shown in Fig.13(b):

$$-iM_{(2)R}^{2(b)} = \frac{i\lambda^2}{(16\pi^2)^2} [\frac{1}{6}(\mu^2 - m^2) - \frac{1}{2}m^2(\ln \frac{\mu^2}{m^2})^2 - \frac{2}{3}m^2 \ln \frac{\mu^2}{m^2} + \frac{1}{12}p^2 \ln \frac{\mu^2}{-p^2}] + \dots \quad (119)$$

We are now in the position to put all the results from diagrams Fig.13(a) and Fig.13(b) together, and obtain the total contributions to the two-loop self-energy,

$$\begin{aligned} -iM_{(2)R}^2 &= \frac{i\lambda^2}{(16\pi^2)^2} [\frac{1}{4}(\mu^2 - m^2) \ln \frac{\mu^2}{m^2} + \frac{1}{6}(\mu^2 - m^2) \\ &\quad - \frac{3}{4}m^2(\ln \frac{\mu^2}{m^2})^2 - \frac{11}{12}m^2 \ln \frac{\mu^2}{m^2} + \frac{1}{12}p^2 \ln \frac{\mu^2}{-p^2}] + \dots \end{aligned} \quad (120)$$

Considering the massless limit  $m^2 \rightarrow 0$  and ignoring the quadratic contribution  $\mu^2 \rightarrow 0$ , one arrives at

$$-iM_{(2)R}^2 = \frac{i\lambda^2}{12(16\pi^2)^2} p^2 \ln \frac{\mu^2}{-p^2}, \quad (121)$$

which agrees with the one obtained by using the standard dimensional regularization method(see page 345 of the book[16]).

### C. Vertex Contribution at Two Loop

The two-loop vertex contribution for the s-channel in  $\phi^4$  theory involves four groups of diagrams as shown in Fig. (15). It is expected that all harmful divergences cancel separately within each group. As the groups of diagrams (d) and (e) are related by a simple interchange of initial and final momenta, they should give the same results and it only needs to calculate either group and then to multiply by a factor of 2.

First, let us calculate the simplest two-loop diagram (f) and its counterterm diagram (f1) in Fig.14. Since the only UV divergence in (f) can be completely canceled by that in (f1), the group (f) does not require two-loop overall counterterm.

$$\begin{aligned} -i\Lambda_{(2)}^{(f)} &= \frac{(-i\lambda)^3}{2} \int \frac{d^4 k_2}{(2\pi)^4} \frac{i}{k_2^2 - m^2} \int \frac{d^4 k_1}{(2\pi)^4} \frac{i^3}{(k_1^2 - m^2)^2 [(k_1 + p)^2 - m^2]} \\ &\rightarrow \frac{i\lambda^3}{2} \frac{-i}{16\pi^2} [M_c^2 - m^2(\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1)] \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int_0^1 dx (1-x) \\ &\quad \int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{\{[(1-x)(k_1^2 - m^2) + x((k_1 + p)^2 - m^2)]^3\}} \\ &= -\frac{i\lambda^3}{2(16\pi^2)^2} [M_c^2 - m^2(\ln \frac{M_c^2}{m^2} - \gamma_\omega + 1)] \int_0^1 dx \frac{1-x}{m^2 - x(1-x)p^2}, \end{aligned} \quad (122)$$

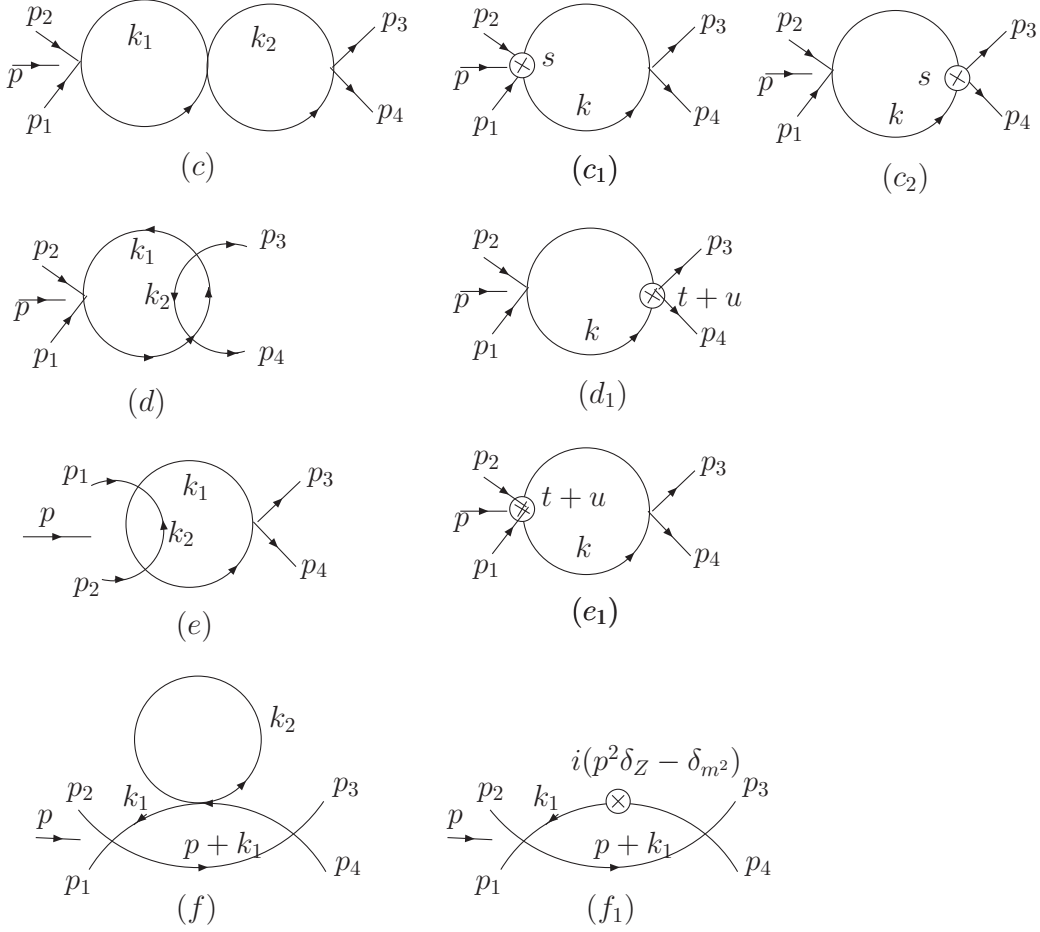


FIG. 15:

$$\begin{aligned}
-i\Lambda_{(2)}^{(f1)} &= i(p^2\delta_Z^{(1)} - \delta_m^{(1)})(-i\lambda)^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{i^3}{(k_1^2 - m^2)^2 [(k_1 + p)^2 - m^2]} \\
&= \frac{i\lambda^3}{2(16\pi^2)^2} [(M_c^2 - \mu^2) - m^2(\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)] \int_0^1 dx \frac{1-x}{m^2 - x(1-x)p^2}, \quad (123)
\end{aligned}$$

$$-i\Lambda_{(2)}^{(f)+(f1)} = -\frac{i\lambda^3}{2(16\pi^2)^2} [(\mu^2 - m^2) - m^2 \ln \frac{\mu^2}{m^2}] \int_0^1 dx \frac{1-x}{m^2 - x(1-x)p^2}. \quad (124)$$

The computation of diagram (c) and its counterterm diagrams (c1) and (c2) in Fig.14 is also straightforward, as these diagrams can be factored into a product of two one-loop integrals. The result is:

$$\begin{aligned}
-i\Lambda_{(2)}^{(c)} &= \frac{1}{4}(-i\lambda)^3 \left[ \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \right]^2 \\
&= \frac{i\lambda^3}{4} \left[ \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx \frac{1}{\{(1-x)(k^2 - m^2) + x[(k+p)^2 - m^2]\}^2} \right]^2 \\
&= -\frac{i\lambda^3}{4(16\pi^2)^2} \left[ \int_0^1 dx (\ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega) \right]^2, \quad (125)
\end{aligned}$$

and

$$\begin{aligned}
-i\Lambda_{(2)}^{(c1)+(c2)} &= 2 \cdot \frac{1}{2}(-i\delta_\lambda^s)(-i\lambda) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \\
&= \frac{i\lambda^3}{2(16\pi^2)^2} (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) [\int_0^1 dx (\ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega)], \quad (126)
\end{aligned}$$

where in the calculation of counterterm diagrams, the factor 2 in the first line accounts for the two equal diagrams (c1) and (c2). By adding up the above results, we have:

$$-i\Lambda_{(2)}^{(c)+(c1)+(c2)} = -\frac{i\lambda^3}{4(16\pi^2)^2} [(\int_0^1 dx \ln \frac{\mu^2}{m^2 - x(1-x)p^2})^2 - (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)^2].$$

By considering the two-loop overall counterterm for diagram (c):

$$-i\delta_\lambda^{(2)(c)} = -\frac{i\lambda^3}{4(16\pi^2)^2} (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)^2, \quad (127)$$

we obtain the renormalized contribution from the group (c) of diagrams:

$$-i\Lambda_{(2)R}^{(c)} = -\frac{i\lambda^3}{4(16\pi^2)^2} [\int_0^1 dx \ln \frac{\mu^2}{m^2 - x(1-x)p^2}]^2. \quad (128)$$

For diagram (d) in Fig.(15), due to the complicated dependence on external momenta, we shall focus on a simplified situation where only the s-channel contributes. Also, we only keep the leading divergent contributions, namely the  $\log \cdot \log$  and  $\log$  terms, because  $\mathcal{M}^2$  can be simplified in such a subdivergence region. To see the momentum dependence in Fig. (15), we write explicitly the expression for diagram (d):

$$\begin{aligned}
-i\Lambda_{(2)}^{(d)} &= \frac{1}{2}(-i\lambda)^3 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2} \frac{i}{(k_1+p)^2 - m^2} \frac{i}{k_2^2 - m^2} \frac{i}{(k_2+k_1+p_3)^2 - m^2} \\
&= \frac{i\lambda^3}{2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int_0^1 dx \frac{1}{[x(k_1+p)^2 + (1-x)k_1^2 - m^2]^2} \cdot \\
&\quad \frac{1}{[k_2^2 - m^2][(k_2+k_1+p_3)^2 - m^2]} \\
&= \frac{i\lambda^3}{2} \int_0^1 dx \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{1}{[k_1^2 + x(1-x)p^2 - m^2]^2} \cdot \\
&\quad \frac{1}{[k_2^2 - m^2][(k_2+k_1+p_3 - xp)^2 - m^2]}. \quad (129)
\end{aligned}$$

Note that in the process to obtain the second and third equalities, we have transformed the original integration into the general  $\alpha\beta\gamma$  integral with  $\alpha = \gamma = 1, \beta = 2$  and  $m_1^2 = m_3^2 = m^2, m_2^2 = m^2 - x(1-x)p^2$  by adopting the usual Feynman parametrization and making the

translation of variable  $k_1 \rightarrow k_1 - xp$ . Thus, the general formulae Eq. (75) gives,

$$\begin{aligned}
-i\Lambda_{(2)}^{(d)} &= -\frac{\lambda^3}{2 \cdot 16\pi^2} \int_0^1 dx \int_0^\infty \prod_{j=1}^3 dv_j \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \\
&\quad \frac{1}{(1+v_2)(3+v_1+v_2+v_3)^2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{[l^2 - \mathcal{M}^2]^2} \\
&\rightarrow -\frac{i\lambda^3}{2 \cdot (16\pi^2)^2} \int_0^1 dx \int_0^\infty \prod_{j=1}^3 dv_j \delta(1 - \sum_{j=1}^3 \frac{1}{1+v_j}) \\
&\quad \frac{1}{(1+v_2)(3+v_1+v_2+v_3)^2} (\ln \frac{M_c^2}{\mathcal{M}^2} - \gamma_\omega), \tag{130}
\end{aligned}$$

with

$$\mathcal{M}^2 = m^2 - \frac{x(1-x)p^2}{1+v_2} - \frac{(p_3 - xp)^2}{3+v_1+v_2+v_3}.$$

In order to carry out the above integral, we transform the UVDP parameters  $v_1, v_2, v_3$  to the new set  $u, v, w$  as shown in Eq. (100), so that the form of  $-i\Lambda_{(2)}^{(d)}$  is changed to:

$$\begin{aligned}
-i\Lambda_{(2)}^{(d)} &= -\frac{i\lambda^3}{2 \cdot (16\pi^2)^2} \int_0^1 dx \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta(1 - \frac{1}{1+w} + \frac{1}{1+v}) \\
&\quad \int_0^\infty du \frac{u}{[u + \frac{1}{(1+w)(1+v)}]^2} [\ln \frac{M_c^2}{\mathcal{M}^2} - \gamma_\omega], \tag{131}
\end{aligned}$$

where  $\mathcal{M}^2$  is given by :

$$\mathcal{M}^2 = m^2 - \frac{u}{u+1} x(1-x)p^2 - \frac{u}{u+1} \cdot \frac{(p_3 - xp)^2}{u(1+w)(1+v)+1}.$$

Notice that in Eq. (131) only the integration over  $u$  is logarithmically divergent, while the ones over  $w, v$  convergent, so we can make the following approximation for  $\mathcal{M}^2$  in the limit  $u \rightarrow \infty$

$$\mathcal{M}^2 \simeq m^2 - x(1-x)p^2, \tag{132}$$

and the integration can be performed as follows

$$\begin{aligned}
-i\Lambda_{(2)}^{(d)} &\simeq -\frac{i\lambda^3}{2 \cdot (16\pi^2)^2} \int_0^1 dx \int_0^\infty \frac{dw}{(1+w)^2} \frac{dv}{(1+v)^2} \delta(1 - \frac{1}{1+w} + \frac{1}{1+v}) \\
&\quad \int_0^\infty du \frac{u}{[u + \frac{1}{(1+w)(1+v)}]^2} [\ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega] \\
&\rightarrow -\frac{i\lambda^3}{2 \cdot (16\pi^2)^2} \int_0^1 dx (\ln \frac{M_c^2}{q_o^2} - \gamma_\omega + 1) (\ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega), \tag{133}
\end{aligned}$$

where the mass scale  $q_o^2$  is taken to be  $m^2 - x(1-x)p^2$  as the only scale in the limit  $u \rightarrow \infty$  is  $\mathcal{M}^2 \simeq m_2^2 = m^2 - x(1-x)p^2$ . Thus we can write down the regularized expression for the diagram (d) as:

$$\begin{aligned}
-i\Lambda_{(2)}^{(d)} &\simeq -\frac{i\lambda^3}{2 \cdot (16\pi^2)^2} \int_0^1 dx [(\ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega)^2 \\
&\quad + (\ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega)]. \tag{134}
\end{aligned}$$

The counterterm diagram ( $d_1$ ) in Fig. 14 can also be simply calculated:

$$\begin{aligned} -i\Lambda_{(2)}^{(d_1)} &= \frac{1}{2}(-i\lambda)(-i\delta_\lambda^{t+u}) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \\ &= \frac{i\lambda^3}{2(16\pi^2)^2} \int_0^1 dx \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) \left( \ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega \right), \end{aligned} \quad (135)$$

where  $\delta_\lambda^{t+u}$  accounts for the  $t$ - and  $u$ - channels in the subdiagram. By combining the above two equations Eq.(134) and (135), we have

$$-i\Lambda_{(2)}^{(d)+(d_1)} \simeq -\frac{i\lambda^3}{2(16\pi^2)^2} \int_0^1 dx \left[ \left( \ln \frac{\mu^2}{m^2 - x(1-x)p^2} \right)^2 + \left( \ln \frac{M_c^2}{m^2 - x(1-x)p^2} - \gamma_\omega \right) \right].$$

Write the two-loop overall counterterm for the diagram (d) as

$$-i\delta_\lambda^{(2)(d)} = \frac{i\lambda^3}{2(16\pi^2)^2} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right), \quad (136)$$

then the two-loop vertex contribution of the diagram (d) is,

$$-i\Lambda_{(2)R}^{(d)} = -\frac{i\lambda^3}{2(16\pi^2)^2} \int_0^1 dx \left[ \left( \ln \frac{\mu^2}{m^2 - x(1-x)p^2} \right)^2 + \ln \frac{\mu^2}{m^2 - x(1-x)p^2} \right] + \dots \quad (137)$$

Obviously, the diagram (e) and its counterterm diagram (e1) in Fig.14 gives the same result as diagram (d) and (d1).

By summing up the renormalized results of the diagrams (c), (d), (e) and (f) in Fig.(15), we finally obtain the s-channel two-loop correction for the four-point function:

$$\begin{aligned} -i\Lambda_{(2)R}^{(s)} &= -\frac{i\lambda^3}{(16\pi^2)^2} \left\{ \frac{1}{4} \left[ \int_0^1 dx \ln \frac{\mu^2}{m^2 - x(1-x)p^2} \right]^2 \right. \\ &\quad + \int_0^1 dx \left[ \left( \ln \frac{\mu^2}{m^2 - x(1-x)p^2} \right)^2 + \ln \frac{\mu^2}{m^2 - x(1-x)p^2} \right] \\ &\quad \left. + \frac{1}{2} [(\mu^2 - m^2) - m^2 \ln \frac{\mu^2}{m^2}] \int_0^1 dx \frac{1-x}{m^2 - x(1-x)p^2} \right\}, \end{aligned} \quad (138)$$

where  $-p^2 = -(p_1 + p_2)^2 \equiv s$  for s-channel.

For completeness, we also have to consider  $t$ - and  $u$ -channel contributions, which will give us the similar expressions except for the substitution of  $p^2$ :  $-p^2 = -(p_1 - p_4)^2 \equiv t$  for  $t$ -channel and  $-p^2 = -(p_1 - p_3)^2 \equiv u$  for  $u$ -channel. In Fig.(16), we only present counterpart of diagram (c) for  $t$ - and  $u$ -channels, and other diagrams in groups (c), (d), (e) and (f) could be obtained by the same change of external legs.

As a consistency check for the above results, let us consider the situation with massless limit  $m \rightarrow 0$  and  $s \rightarrow \infty$  but keeping  $t$  fixed. From the identity  $s + t + u = 0$ , which indicates  $u \simeq -s \rightarrow -\infty$ , one needs to consider  $u$ -channel as well. In this case, the four-point vertex correction is found to be:

$$\begin{aligned} -i\Lambda_{(2)R}^{(s)+(u)} &\simeq -\frac{i\lambda^3}{(16\pi^2)^2} \left[ \frac{5}{4} \left( \ln \frac{\mu^2}{s} \right)^2 + \frac{5}{4} \left( \ln \frac{\mu^2}{u} \right)^2 \right] \\ &\simeq -\frac{5}{2} \cdot \frac{i\lambda^3}{(16\pi^2)^2} \left( \ln \frac{\mu^2}{s} \right)^2 \end{aligned} \quad (139)$$

where in the last equality we only keep the leading external momentum  $s = -p^2 \simeq u$  dependence when the external momentum  $s$  is large. Such a result agrees with the one given in the book[16](on page 345).



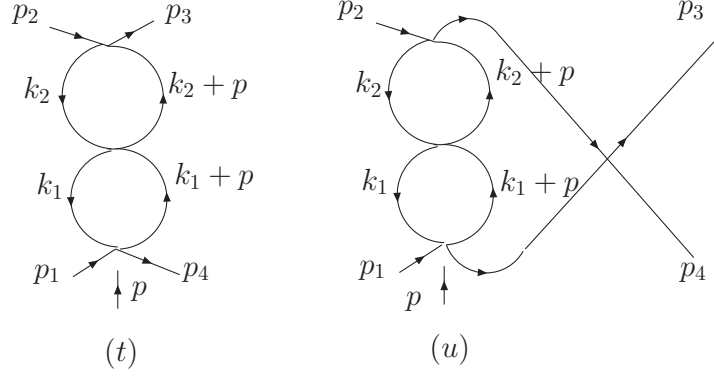


FIG. 16:

#### D. Two Loop $\beta$ Functions and Anomalous Mass Dimension in $\phi^4$ Theory

Having calculated the divergence behavior of all the two-loop diagrams in the  $\phi^4$  theory, we are now ready to obtain two-loop  $\beta$  functions. From Eqs. (96) and (118), we can extract the two-loop mass and wave function counterterms respectively:

$$-i\delta_{m^2}^{(2)} = \frac{i\lambda^2}{(16\pi^2)^2} \left[ \frac{1}{4}(M_c^2 - \mu^2) \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) - \frac{1}{6}(M_c^2 - \mu^2) \right. \\ \left. - \frac{1}{4}m^2 \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right)^2 + \frac{1}{6}m^2 \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) \right], \quad (140)$$

$$i\delta_Z^{(2)} = -\frac{i\lambda^2}{12(16\pi^2)^2} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right). \quad (141)$$

And the two-loop vertex counterterm can be extracted from Eqs. (127) and (136):

$$-i\delta_\lambda^{(2)} = 3 \cdot (-i\delta_\lambda^{(c)} - i\delta_\lambda^{(d)} - i\delta_\lambda^{(e)}) \\ = \frac{i\lambda^3}{(16\pi^2)^2} \left[ -\frac{3}{4} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right)^2 + 3 \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) \right], \quad (142)$$

where the factor 3 in the first line accounts for the s, t, u-channels respectively.

Recall the relation between renormalized coupling constant  $\lambda$  and the bare one  $\lambda_0$  is:

$$\lambda = \lambda_0 Z_\phi^2 - \delta_\lambda = \lambda_0 (1 + \delta_Z)^2 - \delta_\lambda \\ \approx \lambda_0 (1 + 2\delta_Z) - \delta_\lambda, \quad (143)$$

where  $\delta_Z$  and  $\delta_\lambda$  are the function of the bare coupling  $\lambda_0$ , which is independent of scale  $\mu$ . In the perturbative calculation of  $\lambda$  at two-loop level, we have,

$$\lambda \approx \lambda_0 - 2 \cdot \frac{\lambda_0^3}{12(16\pi^2)^2} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) \\ - \frac{3\lambda_0^2}{2 \cdot (16\pi^2)} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) - \frac{\lambda_0^3}{(16\pi^2)^2} \left[ \frac{3}{4} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right)^2 - 3 \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) \right] \\ = \lambda_0 - \frac{3\lambda_0^2}{2 \cdot (16\pi^2)} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) - \frac{\lambda_0^3}{(16\pi^2)^2} \left[ \frac{3}{4} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right)^2 - \frac{17}{6} \left( \ln \frac{M_c^2}{\mu^2} - \gamma_\omega \right) \right]. \quad (144)$$

Thus according to the definition of  $\beta$ -function which is supposed to sum up all the leading logarithmic terms (ignoring the logarithmic-squared term), we arrive at the  $\beta$ -function for the renormalized coupling constant  $\lambda$  as:

$$\begin{aligned}\beta_\lambda &= \mu \frac{d\lambda}{d\mu} \\ &= \frac{3\lambda_0^2}{16\pi^2} - \frac{2\lambda_0^3}{(16\pi^2)^2} \frac{17}{6} \\ &\approx \frac{3\lambda^2}{16\pi^2} - \frac{17}{3} \frac{\lambda^3}{(16\pi^2)^2},\end{aligned}\tag{145}$$

where the bare constant  $\lambda_0$  has been replaced in the last line by its renormalized one, leading to the standard result  $\beta_\lambda$ . [18–20].

Similarly, we can evaluate the anomalous mass dimension at two-loop level. From the definition of the renormalized mass:

$$m^2 = Z_\phi m_0^2 - \delta_{m^2} = m_0^2 + m_0^2 \delta_Z - \delta_{m^2},\tag{146}$$

we have the following approximate relation for the renormalized  $m^2$  given in terms of bare mass  $m_0^2$  and the bare coupling constant  $\lambda_0$  at two-loop level:

$$\begin{aligned}m^2 &= m_0^2 + \frac{\lambda_0}{2(16\pi^2)} [(M_c^2 - \mu^2) - m_0^2 (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)] \\ &\quad + \frac{\lambda_0^2}{(16\pi^2)^2} [\frac{1}{4} (M_c^2 - \mu^2) (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega) - \frac{1}{6} (M_c^2 - \mu^2) \\ &\quad - \frac{1}{4} m_0^2 (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)^2 + \frac{1}{12} m_0^2 (\ln \frac{M_c^2}{\mu^2} - \gamma_\omega)],\end{aligned}\tag{147}$$

which is different from the result obtained by using the dimensional regularization approach due to the appearance of the quadratic terms. The anomalous mass dimension where we sum up all the leading quadratic and logarithmic terms (i.e., not considering the logarithmic-squared term and quadratic-logarithmic cross term) is given by:

$$\begin{aligned}\gamma_{\phi^2} &= \frac{\mu^2}{m^2} \frac{dm^2}{d\mu^2} \\ &= -\frac{\lambda_0}{(16\pi^2)m^2} (\frac{1}{2}\mu^2 - \frac{1}{2}m_0^2) + \frac{\lambda_0^2}{(16\pi^2)^2 m^2} (\frac{1}{6}\mu^2 - \frac{1}{12}m_0^2) \\ &\approx -\frac{\lambda}{16\pi^2} (\frac{1}{2}\frac{\mu^2}{m^2} - \frac{1}{2}) + \frac{\lambda^2}{(16\pi^2)^2} (\frac{1}{6}\frac{\mu^2}{m^2} - \frac{1}{12}) \\ &= \frac{1}{2} \frac{\lambda}{16\pi^2} - \frac{1}{12} \left( \frac{\lambda}{16\pi^2} \right)^2 - \frac{\mu^2}{m^2} \left[ \frac{1}{2} \frac{\lambda}{16\pi^2} - \frac{1}{6} \left( \frac{\lambda}{16\pi^2} \right)^2 \right],\end{aligned}\tag{148}$$

where we have replaced in the third line the bare mass and coupling constant with the renormalized ones.

Note that the resulting  $\gamma_{\phi^2}$  is different from that obtained in ref.[21] by using the dimensional regularization approach with the  $\bar{MS}$  subtraction scheme. The difference occurs in both the power-law and the logarithmic running terms. For the power-law running terms

with the form  $\mu^2/m^2$  in  $\gamma_{\phi^2}$ , it reflects the fact that the LORE method maintains the original quadratic divergence. For the logarithmic terms, the difference can be caused from the well-known fact that the two-loop anomalous mass dimension in  $\phi^4$  theory is in general subtraction scheme dependent. This may be seen from the rescaling  $\mu^2 \rightarrow e^{\alpha_0} \mu^2$ , the resulting leading logarithmic term at two loop level is changed by an additional contribution from the logarithmic-squared term, thus the corresponding  $\gamma_{\phi^2}$  for the logarithmic running is changed to be

$$\gamma_{\phi^2}|_{log} = \frac{1}{2} \frac{\lambda}{16\pi^2} - \frac{1}{12} (1 + 6\alpha_0) \left( \frac{\lambda}{16\pi^2} \right)^2.$$

As a consequence, both the  $\mu^2$ -independent term and the quadratic  $\mu^2$ -dependent terms also changed correspondingly. Similarly, when shifting the scale  $\mu^2 \rightarrow \hat{\mu}^2 \equiv \mu^2 - \alpha_0 m^2$ , the leading logarithmic term also receives an extra contribution from the quadratic-logarithmic cross term, and the resulting  $\gamma_{\phi^2}$  for the logarithmic running in terms of the new subtraction energy scale  $\hat{\mu}^2$  is modified to be

$$\gamma_{\phi^2}|_{log} = \frac{1}{2} \frac{\lambda}{16\pi^2} - \frac{1}{12} (1 + 3\alpha_0) \left( \frac{\lambda}{16\pi^2} \right)^2.$$

However, the quadratic-logarithmic cross term is now given in terms of two energy scales  $\mu^2$  and  $\hat{\mu}^2$  rather than a single one, i.e.,  $(M_c^2 - \mu^2)(\ln M_c^2/\hat{\mu}^2 - \gamma_w)$ . From the above illustration, it is seen that either the rescaling or the shifting of the subtracted energy scale  $\mu^2$  will change the initial correlative form  $(M_c^2 - \mu^2)$  and  $\ln M_c^2/\mu^2$ . Therefore, when the quadratic terms are kept by using the LORE method, the arbitrariness caused by the subtraction scheme for the scalar mass renormalization at high loop order may be eliminated by requiring to maintain the correlative form  $(M_c^2 - \mu^2)$  and  $\ln M_c^2/\mu^2$  with a single subtracted energy scale.

## IX. GENERAL PROCEDURE OF LORE METHOD

With the explicit calculations of two-loop Feynman diagrams in the  $\phi^4$  theory given above, it is useful to summarize the general procedure in applying the LORE method to multi-loop calculations. It is expected that the same procedure is applicable to higher-order calculations with similar features when merging the LORE method with the Bjorken-Drell's analogy between the Feynman diagrams and electrical circuits, though we have only shown it in the two-loop calculations. The procedure may be stated in the following steps:

(i) Write down the corresponding Feynman integrals by using the Feynman rules of the theory for any given Feynman diagrams.

(ii) Combine the denominators by using Feynman parameters to evaluate the two-loop integrals into the sum of the  $\alpha\beta\gamma$  integrals of scalar-type and tensor-type. The use of the usual Feynman parametrization in this step needs to be distinguished from the UVDP parametrization adopted for the  $\alpha\beta\gamma$  integrals. The latter may contain the UV divergences, while the former is in general irrelevant to the UV divergences but it can contain infrared (IR) divergences. From this point of view, making distinction of Feynman parameters from UVDP parameters enables us to separate IR divergences from UV divergences in two parameter spaces.

(iii) By applying the general formulae Eqs.(75-77) for the ILIs of two-loop  $\alpha\beta\gamma$  integrals to the resulting  $\alpha\beta\gamma$  integrals coming from a given Feynman integrals, we can straightforwardly read off the final results for those integrals. Alternatively, one may also adopt a practically useful procedure by completing the squares of the factors in the denominator and evaluate

the  $\alpha\beta\gamma$  integrals into the 2-fold ILIs as proposed in ref.[3], which shows that for each internal loop momentum, one can always transform the integrals into the 1-fold ILIs with respect to it, and then integrate out the ILIs by means of the LORE method. The two procedures are actually equivalent. For tensor-type integrals, we need to apply the consistency conditions Eq. (13) to transform them into the corresponding scalar-type ones first.

(iv) When the integrals involve overlapping divergences, the above procedure will transform the divergences appearing in the subdiagrams into the ones in the UVDP parameter space. In order to identify those divergences, it is helpful to use the advantage of the Bjorken-Drell's analogy between the Feynman diagrams and electric circuits. To extract the UV divergence behavior, it is useful to explore the possible divergence regions in the UVDP parameter space. Then apply the prescription described in Eq.(32) through introducing a mass scale  $q_o^2$  to transform the integrals into the momentum-like ones, so that we can directly apply the LORE method. The scale  $q_o^2$  is in general taken to be the renormalization scale or some intrinsic scales in the original Feynman integrals, such as the masses of particles and/or the external momenta. The explicit form of  $q_o^2$  should be fixed by certain criteria, such as the typical scale in the divergent regions of parameter space, so that the harmful divergences cancel exactly.

It is interesting to notice that in the LORE method only the overall divergence of the overlapping Feynman diagrams is expressed in the momentum integration and the resulting functions  $y_0(x)$  and  $y_2(x)$  can depend on the mass factor  $\mathcal{M}^2$  through which a dependence on kinematic invariants comes in. In contrast, all other divergences arising from the subdiagrams are actually given in terms of the UVDP parameters and the resulting function  $y_0$  in the logarithmic divergence will be independent of any kinematic invariants. This can be seen from the general scalar-type integral Eq.(61) which can arise from n-loop Feynman diagrams. The more detailed evaluations are carried out for the so-called  $\alpha\beta\gamma$  integral of two loop diagrams, which can explicitly be seen in Eqs.(77-84). As a consequence, the regularized divergent quantity of subdiagrams involves only a kinematic-independent scale  $\mu$  via a polynomial of  $\mu^2/M_c^2$  with  $\mu^2/M_c^2 \rightarrow 0$  at  $M_c \rightarrow \infty$ , thus the function  $y_0(x)$  arising from the subdiagrams is no longer a complicated function of kinematic invariants.

## X. CONCLUSIONS AND REMARKS

We have explicitly shown how the loop regularization (LORE) method can be consistently applied to two loop calculations of Feynman diagrams, appropriately treating the overlapping divergences. The key concept of the LORE method[3, 4] is the introduction of the irreducible loop integrals(ILIs), which are generally evaluated from the Feynman diagrams by using the Feynman parametrization and the ultraviolet-divergence-preserving(UVDP) parametrization. We have demonstrated in this paper how the evaluation of ILIs and UVDP parametrization naturally merges with the Bjorken-Drell's analogy between Feynman diagrams and electric circuits. In particular, the UVDP parameters can be regarded as the conductance or resistance in the electric circuit analogy, and the sets of conditions required for evaluating the ILIs and the momentum conservations have been found to associate with the conservations of electric voltages in each loop and the conservations of electric currents at each vertex respectively. As a consequence, the divergences in Feynman diagrams correspond to infinite conductances or zero resistances in electric circuits, and the LORE method merging with the Bjorken-Drell's analogy has the advantage in analyzing the complicated overlapping divergence structure of Feynman diagrams. Therefore, the Bjorken-Drell's circuit analogy allows us to clarify the origin of UV divergences in the UVDP parameter space

and identify the correspondence of the divergences between subdiagrams and UVDP parameters. From the explicit calculations of the case with  $\alpha = \beta = \gamma = 1$  in the general  $\alpha\beta\gamma$  integral, the divergences arising from the subintegrals manifest themselves in the integration over the corresponding asymptotic regions of the UVDP parameter space. The calculations of the corresponding counterterm diagrams confirm our intuitive picture that all the harmful divergences cancel exactly in the final result. Although the procedures and calculations in the LORE method are not as concise as the ones in the dimensional regularization, the overlapping divergent structure and behavior as well as its treatment become more physically clear in the LORE method.

As an interesting application, we have taken the massive scalar  $\phi^4$  theory as an example and performed the detailed calculation of two loop contributions by applying the general formalism of the LORE method. By explicitly computing the two- and four-point functions at two-loop level and carefully using the advantage of Bjorken-Drell's circuit analogy, all the harmful divergences cancel exactly and the resulting two loop corrections agree with the standard results for the logarithmic corrections. The power-law running of mass is explicitly given at two loop level.

In this paper, we have only carried out two-loop calculations and explicitly demonstrated the consistency of the LORE method at two-loop level, but it can be shown that the general procedure of the LORE method shown in Eqs.(57-61) is applicable to even higher-loop calculations by taking advantage of Bjorken-Drell's circuit analogy. Furthermore, we only considered the scalar-type two-loop integrals. However, as shown in [4], in order to ensure the gauge invariance, it is necessary to keep the consistency conditions Eq.(13) which correctly transform the tensor-type ILIs into the scalar-type ones. We shall demonstrate how these consistency conditions in two-loop or even higher-loop order by an explicit calculation[22], although it has already been demonstrated in a general way in[4]. We would like to point out that the advantage of merging the LORE method with Bjorken-Drell's circuit analogy enables us to figure out a more general and rigorous proof for the validity of the LORE method to all orders in the perturbation theory[23].

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## Appendix A: Useful Formulae in UV Divergence Preserving (UVDP) Parametrization

The introduction of UVDP parameters is to combine the various denominators propagating factors, whose utility is similar to Feynman parameters. The motivation to introduce a new UVDP parametrization method is to transform a divergent integral in the UVDP parameter space into a ILI-like divergent one, the object regularized by the LORE method. The simplest case is to combine only two factors in the denominator by using the identity:

$$\frac{1}{AB} = \int_0^\infty \frac{du}{(1+u)^2} \frac{dv}{(1+v)^2} \delta\left(1 - \frac{1}{1+u} - \frac{1}{1+v}\right) \frac{1}{\left[\frac{A}{1+u} + \frac{B}{1+v}\right]^2}.$$

If one of the factors have more than one power, we can differentiate with respect to  $A$  or  $B$  to get,

$$\frac{1}{AB^n} = \int_0^\infty \frac{du}{(1+u)^2} \frac{dv}{(1+v)^2} \delta\left(1 - \frac{1}{1+u} - \frac{1}{1+v}\right) \frac{\frac{n}{(1+v)^{n-1}}}{[\frac{A}{1+u} + \frac{B}{1+v}]^{n+1}}.$$

More general identity for more than two factors is:

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^2} \delta\left(\sum_{i=1}^n \frac{1}{1+v_i} - 1\right) \frac{(n-1)!}{[\sum_{i=1}^n \frac{A_i}{1+v_i}]^n}.$$

Even more general form can be derived:

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \int_0^\infty \prod_{i=1}^n \frac{dv_i}{(1+v_i)^2} \delta\left(\sum_{i=1}^n \frac{1}{1+v_i} - 1\right) \frac{\prod_{i=1}^n \frac{1}{(1+v_i)^{m_i-1}}}{[\sum_{i=1}^n \frac{A_i}{1+v_i}]^{\sum_{i=1}^n m_i}}. \quad (\text{A1})$$

Alternatively, we may also take another more useful form for the case of two factors by just integrating out one of the parameters  $u$  and  $v$  by using the delta function, which has been adopted in [3]:

$$\frac{1}{AB^n} = \int_0^\infty du \frac{nu^{n-1}}{[A + uB]^{n+1}}, \quad (\text{A2})$$

but this form cannot be generalized to the more general case easily.

From the general identity Eq.(A1), we notice that the relation of the UVDP parameters  $v_i$  to Feynman parameters  $x_i$  is:

$$x_i = \frac{1}{1+v_i}. \quad (\text{A3})$$

This identification allows us to transform a divergent integral with Feynman parameters into the one with UVDP parameters, which can be further transformed into a ILI-like integral by introducing a free mass scale and being regularized in the framework of the LORE method. Such a trick is discussed in Eq.(32).

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